Higher-Rank Fields and Currents

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Plan

I $Sp(2M)$ **invariant space: arbitrary rank fields and equations**

II $Sp(4)$ **invariant space:** $3d$ **fields, equations and currents**

III σ_-^{r}**-cohomology analysis and homotopy equation**

IV **South-West principle**

V **Higher** σ_-^{r}**-cohomologies**

VI **Conclusion**

$Sp(2M)$ invariant space

Rank-one unfolded equation

Fronsdal (1986)

Bandos, Lukierski, Sorokin (2000)

$$\left(\xi^{AB}\frac{\partial}{\partial X^{AB}} \pm i\sigma_-\right) C^{\pm}(Y|X) = 0\,, \qquad \sigma_- = \xi^{AB}\frac{\partial^2}{\partial Y^A \partial Y^B}\,,$$

X^{AB} **matrix coordinates of** $\mathcal{M}_M$, $X^{AB} = X^{BA}$ $(A, B = 1, \dots, M)$

Y^A - **auxiliary commuting variables = twistor variables** abusing terminology

$\xi^{MN} = dX^{MN}$ - **anti-commuting variables** $\xi^{MN} = \xi^{NM}\,, \xi^{MN}\xi^{AD} = -\xi^{AD}\xi^{MN}$.

Rank-one primary (dynamical) fields : $\sigma_- C(X|Y) = 0\,:\quad C(X)\,,\ C_A(X)Y^A$

Unfolded equations $\Rightarrow$ **dynamical equations**

$$\frac{\partial}{\partial X^{AE}}\frac{\partial}{\partial X^{BD}}C(X) - \frac{\partial}{\partial X^{BE}}\frac{\partial}{\partial X^{AD}}C(X) \;=\; 0 \qquad \text{Klein-Gordon–like}$$

$$\frac{\partial}{\partial X^{BD}}C_A(X) - \frac{\partial}{\partial X^{AD}}C_B(X) \;=\; 0 \qquad \text{Dirac–like}$$

Rank-r dynamical fields in $\mathcal{M}_M$

Rank- $\mathbf{r}$ unfolded equations : $\mathbf{r}$ sets of twistor variables Y,

$$\Big(\xi^{AB}\frac{\partial}{\partial X^{AB}} \pm i\sigma_-^{\mathbf{r}}\Big)\, C^{\pm}(Y|X) = 0 \ ,$$

$$\sigma_-^{\mathbf{r}} = \xi^{AB} \sum_{j=1}^{\mathbf{r}} \frac{\partial^2}{\partial Y_j^A \partial Y_i^B}\, \delta_{ij}\,, \qquad i, j, \ldots = 1, \ldots, \mathbf{r} \ \text{-color indices}$$

Rank-$\mathbf{r}$ primary fields : $\sigma_-^{\mathbf{r}}\, C(Y|X) = 0 \quad \Rightarrow$

$$C(Y|X) = \sum_n C^{i_1;\ldots;i_n}_{A_1;\ldots;A_n}(X) Y_{i_1}^{A_1} \cdots Y_{i_n}^{A_n} \Rightarrow \text{tracelessness: } \delta_{\mathbf{i_1 i_2}} C^{\mathbf{i_1};\mathbf{i_2};\ldots}_{\ldots}(X) = 0.$$

$$Y_i^A \quad \text{commute} \ \Rightarrow \quad C^{\ldots\mathbf{i_m}\ldots\ \mathbf{i_k}\ldots}_{\ldots\mathbf{A_m}\ldots\mathbf{A_k}\ldots}(X) = C^{\ldots\mathbf{i_k}\ldots\ \mathbf{i_m}\ldots}_{\ldots\mathbf{A_k}\ldots\mathbf{A_m}\ldots}(X) \Rightarrow$$

rank-$\mathbf{r}$ primary fields – tensors $C_{\mathbf{Y}_0}(Y|X)$ described by traceless $\mathfrak{gl}_M$ Young diagrams $\mathbf{Y}_0[h_1, \ldots]$ with respect to Latin indices $A, B = 1, \ldots, M$, *i.e.,* $h_1 + h_2 \leq \mathbf{r}$, $\qquad h_1 \leq M.$

σ_--cohomology analysis: Rank-$\mathbf{r}$ primary fields and field equations are represented by the cohomology groups $H^0(\sigma_-^{\mathbf{r}})$ and $H^1(\sigma_-^{\mathbf{r}})$, respectively.

Shaynkman, Vasiliev (1989)

Result: Rank-$\mathbf{r}$ dynamical equations

Rank-$\mathbf{r}$ primary fields $C_{\mathbf{Y}_0}(Y|X)$ satisfy rank-$\mathbf{r}$ dynamical equations

$$\mathcal{E}^{A_1[\mathbf{r}-h_2+1]\,,\,A_2[\mathbf{r}-h_1+1]\,,\,A_3[h_3],\ldots,A_n[h_n]}_{i_1[h_1]\,,\qquad i_2[h_2]\,,\qquad i_3[h_3],\ldots,\,i_n[h_n]}\underbrace{\frac{\partial}{\partial Y^{A_1^1}_{i_1^1}}\cdots\frac{\partial}{\partial Y^{A_1^{h_1}}_{i_1^{h_1}}}}_{h_1}\cdots\underbrace{\frac{\partial}{\partial Y^{A_n^1}_{i_n^1}}\cdots\frac{\partial}{\partial Y^{A_n^{h_n}}_{i_n^{h_n}}}}_{h_n}$$

$$\underbrace{\frac{\partial}{\partial X^{A_1^{h_1+1}\;A_2^{h_2+1}}}\cdots\frac{\partial}{\partial X^{A_1^{\mathbf{r}-h_2+1}\;A_2^{\mathbf{r}-h_1+1}}}}_{\mathbf{r}+1-h_1-h_2}C_{\mathbf{Y}_0}(Y|X)=0$$

The symmetry properties of the parameter $\mathcal{E}^{\cdots}_{\cdots}$ described by

$$\mathbf{Y}_0[h_1,h_2,h_3,\ldots,h_n]$$ with respect to the lower indices

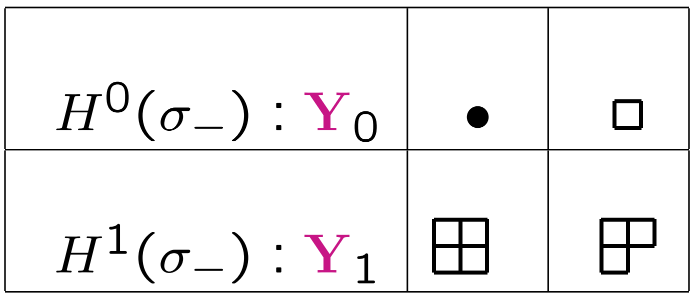
and by its rank-$\mathbf{r}$ two-column dual

$$\mathbf{Y}_1[\mathbf{r}+1-h_2,\mathbf{r}+1-h_1,h_3,\ldots,h_n]$$

with respect to the upper ones.

Examples

The full lists of YD associated with rank-1 and rank-2 fields and equations :



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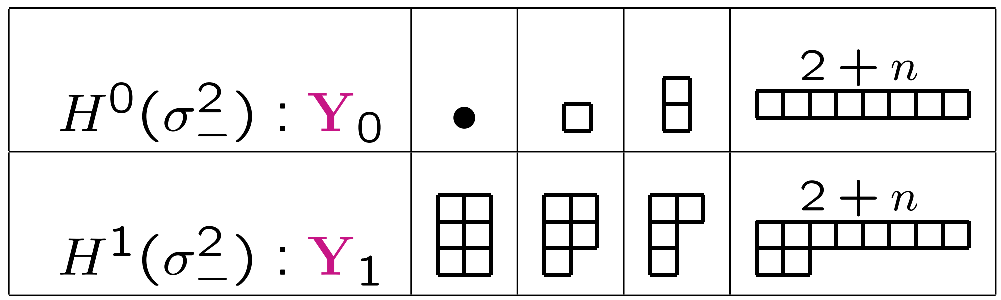

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Two-column duality

$3d$ conformal fields and equations in $Sp(4)$ invariant space

Free $3d$ massless fields $C(t,x)$ can be described in terms of

two-component spinors y^α and symmetric matrix

$$x^{\alpha\beta} = x^{\beta\alpha}: \quad x^{\alpha\beta} = t\delta^{\alpha\beta} + x^1\sigma_1^{\alpha\beta} + x^2\sigma_3^{\alpha\beta} \quad , \qquad \alpha,\beta = 1,2\,,$$

where $\sigma_{1,3}^{\alpha\beta}$ – traceless symmetric Pauli matrices.

Conformal invariant massless equations $=$ Rank-1 unfolded equations

Shaynkman, Vasiliev (2001)

$$dx^{\alpha\beta}\left(\frac{\partial}{\partial x^{\alpha\beta}} \pm i\frac{\partial^2}{\partial y^\alpha \partial y^\beta}\right) C^\pm(y|x) = 0 \qquad \Rightarrow$$

primaries : $\dfrac{\partial^2}{\partial y^\alpha \partial y^\beta} b(x) = \dfrac{\partial^2}{\partial y^\alpha \partial y^\beta} f_\beta(x) y^\beta = 0$

boson $b(x)$: $\varepsilon^{\beta\nu}\epsilon^{\alpha\gamma}\dfrac{\partial^2}{\partial x^{\alpha\beta}\partial x^{\gamma\nu}} b(x) = 0$ $\sim$ $3d$ Klein-Gordon

fermion $f_\beta(x)$: $\epsilon^{\alpha\gamma}\dfrac{\partial}{\partial x^{\alpha\beta}} f_\gamma(x) = 0$ $\sim$ $3d$ Dirac

$\epsilon^{\alpha\beta}$ - 2×2 symplectic form .

$3d$ conformal currents

Rank-2 equations $=$ **Current equations** O.G, M.Vasiliev (2003)

$$dx^{\alpha\beta}\left\{\frac{\partial}{\partial x^{\alpha\beta}} - i\frac{\partial^2}{\partial v^\alpha \partial u^\beta}\right\}\mathcal{J}(u,\, v|x) = 0\,, \qquad v = \frac{1}{2}(y_1 + y_2),\;\; u = \frac{1}{2}(y_1 - y_2)$$

Closed differential forms $=$ $3d$ **conserved current**

$$\left(i\,dx^{\alpha\beta}\frac{\partial}{\partial u^\beta} + d\,v^\alpha\right)^2 \mathcal{J}(u\,,v|x)\Big|_{u=0}$$

Current equations are obeyed by generalized bilinear stress tensors

$$\mathcal{J} = T^{k\,l}_{\alpha_1\ldots\alpha_n}(x) = \frac{\partial}{\partial u^{\alpha_1}}\cdots\frac{\partial}{\partial u^{\alpha_n}}\Big(C^k_+(v-u|x)C^l_-(v+u|x)\Big)|_{u=0}$$ **:**

$C_\pm(y|x)$ **– rank-1 fields.**

$u \leftrightarrow v$:

$$\left(i\,dx^{\alpha\beta}\frac{\partial}{\partial v^\beta} + d\,u^\alpha\right)^2 \widetilde{\mathcal{J}}(u\,,v|x)\Big|_{v=0}\,,$$

$$\widetilde{\mathcal{J}} = \tilde{T}^{k\,l}_{\alpha_1\ldots\alpha_n}(x) = \frac{\partial}{\partial v^{\alpha_1}}\cdots\frac{\partial}{\partial v^{\alpha_n}}\Big(C^k_+(v-u|x)C^l_-(v+u|x)\Big)|_{v=0}$$

$H^N(\sigma_-^{\mathbf{r}})$: polynomials $P(Y,\xi)$

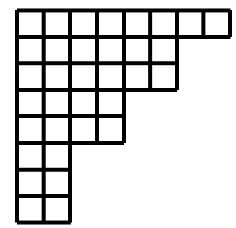
$\mathfrak{o}(\mathbf{r})$-module with respect to the color indices i carried by Y_i^A

$\mathfrak{gl}_M$- module with respect to the spinor indices $A, B, \ldots$ of Y_i^A and ξ^{AB}

$\mathfrak{o}(\mathbf{r})$-module characterized by a traceless $\mathbf{Y}_0$ and a traceful part,

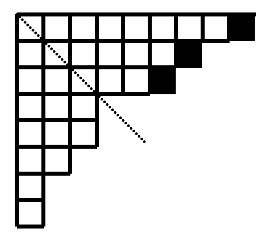
composed of the Kronecker symbols δ^{ij} : Kronecker YD $\mathbf{Y}_\delta$

$\mathbf{Y}_\delta \in$ symmetrized $\left(\otimes^K \square\!\square\right)$:



$(\xi)^N$: Almost symmetric Young diagrams $\mathbf{Y}_A^N$ with $2N$ cells

$\mathbf{Y}_A^N \in$ anti-symmetrized $\left[\otimes^N \square\!\square\right]$:



Given $\mathbf{Y}_0$, $\mathbf{Y}_A^N$ $\Rightarrow$ $\exists\ \mathbf{Y}_\delta$: $\exists \mathbf{Y}_H \in \mathbf{Y}_0 \otimes \mathbf{Y}_\delta \otimes \mathbf{Y}_A^N$ represents $H^N(\sigma_-^{\mathbf{r}})$

M is large enough: does not restrict $\mathfrak{gl}_M$-Young diagrams

Homotopy operator

Standard homotopy trick : Conjugated linear operators

Ω and Ω^* , $\Omega^2 = 0 \Rightarrow \Delta = \{\Omega, \Omega^*\}$ - **semi-positive homotopy operator**

If Δ is diagonalizable $\Rightarrow H(\Omega) \subset \ker\,\Delta \cap \ker\,\Omega\,.$

$$\Omega := \sigma_-^{\mathbf{r}} = T_{AB}\xi^{AB}\,, \qquad \Omega^* = T^{AB}\frac{\partial}{\partial \xi^{CD}}\,,$$

$$T_{AB} = \frac{\partial}{\partial Y_i^A}\frac{\partial}{\partial Y_j^B}\delta^{ij}, \quad T^{CD} = Y_i^C Y_j^D \delta^{ij}\,, \quad T_B^A = Y_j^A \frac{\partial}{\partial Y_j^B} \qquad = \mathfrak{sp}(2M)$$

$$\boxed{\Delta = \{\Omega\,,\Omega^*\} = \tfrac{1}{2}\tau_{mk}\tau^{mk} + \nu_B^A\nu_A^B - (M+1-\mathbf{r})\nu_A^A}$$

$\tau_{mk} = Y_m{}^A \dfrac{\partial}{\partial Y^{kA}} - Y_k{}^A \dfrac{\partial}{\partial Y^{mA}}$ – **generators of** $\mathfrak{o}(\mathbf{r})$,

$\nu_B^A = \chi_B^A + T_B^A$ – **generators of** $\mathfrak{gl}_M^{tot}$ **that acts on** Y_i^A **and** ξ^{AB}

$\chi_B^A = 2\xi^{AD}\dfrac{\partial}{\partial \xi^{BD}}$ – **generators of** $\mathfrak{gl}_M$

Young diagrams and Casimir operators

$$\mathbf{Y}_H[B_1,\ldots] \in \mathbf{Y}_0[h_1,\ldots] \otimes \mathbf{Y}_\delta[d_1,d_1,\ldots] \otimes \mathbf{Y}_A[a_1,\ldots]$$

Casimir operators:

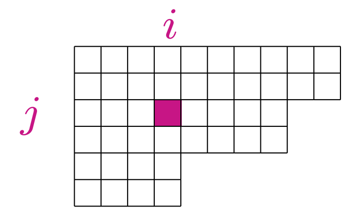
$$\begin{aligned} \tau_{mk}\tau^{mk} &= 2\sum_j h_j(h_j - \mathbf{r} - 2(i-1))\,, \\ \nu^A_B \nu^B_A &= -\sum_i B_i(B_i - M - 1 - 2(i-1)) \end{aligned}$$

$\Rightarrow$ **Homotopy operator**

$$\Delta = -\sum_i B_i(B_i - 2(i-1)) + \sum_j h_i(h_i - 2(i-1)) + \mathbf{r}\sum_i (B_i - h_i)$$

South-West principle

$\mathcal{S}(i,j)$ – a cell on the intersection of $j-th$ row and $i-th$ column



$$\mathbf{Y} = \bigcup_{\mathcal{S}(i,j)\in\mathbf{Y}} \mathcal{S}(i,j)$$

Numerical characteristic $\chi^a(\mathcal{S}(i,j)) = i - j + a\,,$ $\qquad \chi^a(\mathbf{Y}) = \sum_{\mathcal{S}\in\mathbf{Y}} \chi^a(\mathcal{S}) \quad \forall a$

$\chi^a(\mathbf{Y}) = -\frac{1}{2}\sum_i h_i(h_i - 2i + 1 - 2a)$ since $\mathbf{Y}[h_1,\ldots]$ = unification of columns

$$\Rightarrow \quad \Delta = \chi^{\frac{1}{2}(\mathbf{r}-1)}(\mathbf{Y}_H \setminus \mathbf{Y}_0)$$

$\mathcal{S}_1$ is situated the more South-West then $\mathcal{S}_2 \quad \Leftrightarrow \quad \chi^a(\mathcal{S}_1) < \chi^a(\mathcal{S}_2)$

Δ semi-positive $\quad \Rightarrow \quad \min(\Delta)$ is reached when all cells of $\mathbf{Y}_H$

are maximally South-West . It allows us to find $H^N(\sigma_-^{\mathbf{r}})\ \forall N$

Homotopy equation

$$\chi^{\frac{1}{2}(\mathbf{r}-1)}(\mathbf{Y}_H \setminus \mathbf{Y}_0) = 0$$

Higher $\sigma_-^{\mathbf{r}}$- cohomologies in $\mathcal{M}_M$: Result

The full list of YD associated with $H^N(\sigma_-^{\mathbf{r}})$:

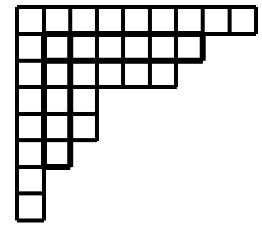
$$\mathbf{Y}_H\,[B_1, B_2, \ldots] : \quad B_j = h_j + a_j + \sum_j \mathbf{s}_{i\ j}(\mathbf{Y}_0, \mathbf{Y}_A)$$

arbitrary $\mathbf{Y}_0[h_1, \ldots]$ – traceless YD

arbitrary $\mathbf{Y}_A[a_1, \ldots]$ – almost symmetric YD : $\sum a_i = 2N$

resulting $\mathrm{s}(\mathbf{Y}_0, \mathbf{Y}_A)$ – shift matrix inherits the structure of $\mathbf{Y}_A$

resulting $\mathbf{Y}_\delta(\mathbf{Y}_0, \mathbf{Y}_A)$ – Kronecker YD

Example: $\mathbf{Y}_A[8,6,5,3,3,3,2,1,1] =$  $\Rightarrow$

$$
\mathbf{s}(\mathbf{Y}_0, \mathbf{Y}_A) = \begin{pmatrix}
\Delta_0 & \Delta_0 & \Delta_2 & \Delta_3 & \Delta_4 & \Delta_5 & \Delta_6 & \Delta_7 & \Delta_8 \\
\Delta_2 & \Delta_1 & \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 & \Delta_5 & 0 & 0 \\
\Delta_3 & \Delta_2 & \Delta_0 & \Delta_0 & \Delta_2 & \Delta_3 & 0 & 0 & 0 \\
\Delta_4 & \Delta_3 & \Delta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Delta_5 & \Delta_4 & \Delta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Delta_6 & \Delta_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Delta_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Delta_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$\Delta_0 = \mathbf{r} - h_1 - h_2$$

$$\Delta_k = h_k - h_{k+1} \quad (k > 0)\,.$$

Kronecker YD $\mathbf{Y}_\delta[d_1, d_1, d_2, d_2, d_3, d_3]$:

$$d_1 = \Delta_0 + \Delta_2 + \cdots + \Delta_8\,, \quad d_2 = \Delta_1 + \Delta_2 + \Delta_4 + \Delta_5\,, \quad d_3 = \Delta_0 + \Delta_2 + \Delta_3$$

Odd and even "nested hooks" are different

Minkowski-like reduction.

$4d$ Minkowski unfolded equations is a subsystem of rank-2 ones in $\mathcal{M}_4$

with $Y^A = (y^\alpha,\ \bar{y}^{\beta'})$, $\qquad X^{AB} = (x^{\alpha\beta'},\ x^{\alpha\beta},\ \bar{x}^{\alpha'\beta'})$ $\quad (\ \overline{y^\alpha} = \bar{y}^{\alpha'})$.

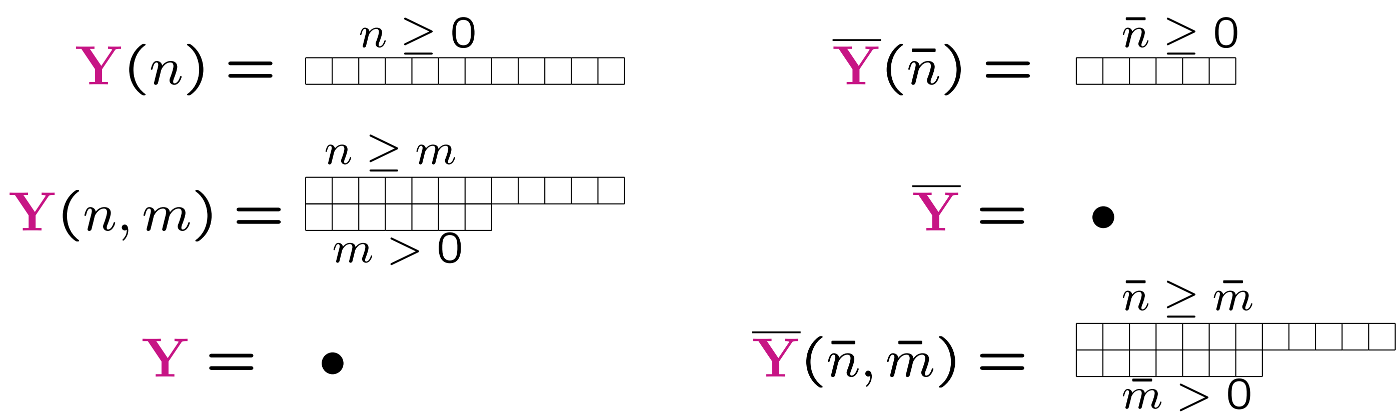
Rank-$\mathbf{r}$ primary fields $C_{\mathbf{Y}\,,\overline{\mathbf{Y}}}(y,\bar{y}|x)$ are described by

pairs of the mutually traceless Young diagrams

$\mathbf{Y}[h_1,\ldots,h_k]$ and $\overline{\mathbf{Y}}[\bar{h}_1,\ldots,\bar{h}_n]$: $h_1+\bar{h}_1 \le \mathbf{r}$.

For example, Minkowsky primary free currents =rank-2 Minkowsky

primary fields are described by



Conclusion

All higher $\sigma_-^{\mathbf{r}}$ cohomologies are found $\Rightarrow$

All primary fields, associated with differential forms of different degrees, and their equations are found, including usual $3d$ and $4d$ any rank Minkowsky fields and field equations

Applications to the construction of interacting theories