Conformal higher-spin fields in (super) hyperspace

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Motivation

- Different formulations of a theory are useful for revealing its different properties and features
  - Metric-like formulation *(Fierz & Pauli '39,…Weinberg '64,…Fronsdal '78,…)*

$$g_{mn}(x) \quad \Rightarrow \quad \varphi_{m_1\cdots m_s}(x)\,, \qquad \delta\varphi_{m_1\cdots m_s} = \partial_{(m_1}\xi_{m_2\cdots m_s)}(x)$$

$$R^{(s)} = \partial^s \varphi^{(s)}(x)$$ - Higher spin curvatures

  - Frame-like formulation *(Vasiliev '80, Aragone & Deser '80 …)*

$$dx^m e_m^a,\;\; dx^m \omega_m^{ab} \;\;\Rightarrow\;\; dx^m e_m^{a_1\ldots a_{s-1}},\;\; dx^m \omega_m^{a_1\ldots a_{s-1},b_1\ldots b_t} \quad (t=1,\ldots,s-1), \quad R^{(s)} = d\omega^{s-1,s-1}$$

- Unfolded HS dynamics

- Vasiliev non-linear HS equations involve infinite number of fields

4d HS fields: $\omega(x,y,\bar{y}) = \sum_{k,j=0}^{\infty} dx^m \omega_m^{\alpha_1\ldots\alpha_k,\dot\beta_1\ldots\dot\beta_j}(x)\, y_{\alpha_1}\ldots y_{\alpha_k}\bar{y}_{\dot\beta_1}\ldots\bar{y}_{\dot\beta_k}$

extension of 4d space-time with spinorial (twistor-like) directions

Motivation

We will be interested in a different hyperspace extension which also incorporates all 4d HS fields - an alternative to Kaluza-Klein.

Free HS theory is a simple theory of a "hyper" scalar and spinor which posses an extended conformal symmetry

- Study of (hidden) symmetries can provide deeper insights into the structure of the theory and may help to find its most appropriated description
  - HS symmetries of HS theory are infinite-dimensional. They control in a very restrictive way the form of the non-linear equations and, hence, HS field interactions

- Question: what is the largest **finite-dimensional** symmetry of a HS system and can we learn something new from it?
  - extended conformal symmetry
  - the description of the HS system with CFT methods

Sp(8) symmetry of 4d HS theory *(Fronsdal, 1985)*

$$SO(1,3) \subset SO(2,3) \subset SO(2,4) \subset Sp(8,\mathrm{R})$$
$$SO(2,3) \approx Sp(4,\mathrm{R})$$

Sp(8) acts on infinite spectrum of 4d HS single-particle states (s=0, 1/2, 1, 3/2, 2,…)

this is a consequence of Flato-Fronsdal Theorem, *1978*:

Tensor product of two **3d singleton** moduli comprises all the massless spin-s fields in 4d

Singletons are 3d massless scalar and spinor fields which enjoy 3d conformal symmetry

$$SO(2,3) \approx Sp(4)$$

$$\mathrm{S} \otimes \mathrm{S} \quad \Rightarrow \quad Sp(4) \times Sp(4) \subset Sp(8)$$

Can Sp(8) play a role similar to Poincaré or conformal symmetry acting **geometrically** on a hyperspace containing 4d space-time?

Whether a 4d HS theory can be formulated as a field theory on this hyperspace?

`Geometrically' means: $\delta_{conf}\, x^m = a^m + l^m{}_n x^n + b x^m + k^m x^2 - 2k_n x^n x^m$

Fronsdal '85:

minimal dimension of the Sp(8) hyperspace (containing 4d space-time) is 10

Particles and fields in Sp(8) hyperspace

- It took about 15 years to realize Fronsdal's idea in concrete terms

Bandos & Lukierski '98: Twistor-like (super) particle on a tensorial space (their motivation was not related to HS theory, but to supersymmetry)

Most general $N=1$ susy in flat 4d: $\{Q_\alpha, Q_\beta\} = \gamma^m_{\alpha\beta} P_m + \gamma^{mn}_{\alpha\beta} Z_{mn}, \quad [P_m, Z_{nl}] = 0$

$$\gamma^m_{\alpha\beta} = \gamma^m_{\beta\alpha}, \quad \alpha, \beta = 1,2,3,4$$

10d space coordinates: $P_m \to x^m$ (4d coordinates), $Z_{mn} \to y^{mn} = -y^{nm}$ (6 extra coordinates)

$X^{\alpha\beta} = X^{\beta\alpha} = \frac{1}{2} x^m \gamma^{\alpha\beta}_m + \frac{1}{4} y^{mn} \gamma^{\alpha\beta}_{mn}$ - 4×4 matrix coordinates

Superparticle action: $S = \int d\tau\, \lambda_\alpha \lambda_\beta (\dot{X}^{\alpha\beta} - i\theta^\alpha \dot{\theta}^\beta), \quad \lambda_\alpha$ - commuting twistor-like variable

possesses hidden (generalized superconformal) symmetry $OSp(1|8) \supset Sp(8)$

Quantization (*Bandos, Lukierski & D.S. '99*)

$$\left(\frac{\partial}{\partial X^{\alpha\beta}} - i\lambda_\alpha \lambda_\beta\right)\Phi(X, \lambda) = 0$$

- describes in 4d free fields of any spin s=0, 1/2, 1, 3/2, 2, …

- *P. West '07* E(11) and Higher Spin Theories in M-theory

Field theory in flat Sp(8) hyperspace

Field equations in flat hyperspace *(Vasiliev '01)*:

Forier transform $C(X,\xi)=\int d^4\lambda\, e^{i\lambda_\alpha\xi^\alpha}\Phi(X,\lambda) \Rightarrow \left(\frac{\partial}{\partial X^{\alpha\beta}}+i\frac{\partial^2}{\partial\xi^\alpha\partial\xi^\beta}\right)C(X,\xi)=0$ Free unfolded equations

$$C(X,\xi)=b(X)+\xi^\alpha f_\alpha(X)+\sum \xi^{\alpha_1}\ldots\xi^{\alpha_k}C_{\alpha_1\ldots\alpha_k}(X)$$

$b(X)$ and $f_\alpha(X)$ are independent scalar and spinor hyperfields satisfying the equations:

$$(\partial_{\alpha\beta}\partial_{\gamma\delta}-\partial_{\alpha\gamma}\partial_{\beta\delta})\,b(X)=0$$
$$\partial_{\alpha\beta}f_\gamma(X)-\partial_{\alpha\gamma}f_\beta(X)=0$$

Section condition in generalized geometry of M-theory *(Berman et. al.)*

$$\supset \quad \frac{\partial^2 b(x,y)}{\partial x_m\partial y^{mn}}=0, \qquad \varepsilon^{mnlp}\frac{\partial^2 b(x,y)}{\partial y^{mn}\partial y^{lp}}=0$$

4d content of $b(X)$ and $f_\alpha(X)$ are Higher-Spin curvatures (*Vasiliev '01, Bandos et. al. '05*):

$(X^{\alpha\beta}=X^{\beta\alpha}=\frac{1}{2}x^m\gamma_m^{\alpha\beta}+\frac{1}{4}y^{mn}\gamma_{mn}^{\alpha\beta})$

Integer spins: $b(x^m,y^{mn})=\varphi(x)+F_{mn}(x)y^{mn}+(R_{mn,pq}(x)-\frac{1}{2}\eta_{mp}\partial_n\partial_q\varphi)y^{mn}y^{pq}+\ldots$

½ integer spins: $f^\alpha(x^m,y^{mn})=\psi^\alpha(x)+\left(\Psi_{mn}^\alpha(x)-\frac{1}{2}\partial_m(\gamma_n\psi)^\alpha\right)y^{mn}+\ldots$

Eoms and Bianchi: $\partial^2\varphi=0;\quad \partial_{[l}F_{mn]}=0,\quad \partial^m F_{mn}=0;\quad R_{[mn,p]q}=0,\quad \eta^{mp}R_{mn,pq}=0;\quad \ldots.$

Sp(8) transformations in hyperspace

(symmetries of the field equations)

Conformal transformations: $\delta_{conf}\, x^m = a^m + l^m{}_n x^n + b x^m + k^m x^2 - 2k_n x^n x^m$

Sp(8) transformations: $\delta_{Sp(8)}\, X^{\alpha\beta} = a^{\alpha\beta} + 2g^{(\alpha}{}_{\gamma}\, X^{\beta)\gamma} - X^{\alpha\gamma} k_{\gamma\delta} X^{\delta\beta}$

$$\delta\, b = -\delta\, X^{\alpha\beta}\partial_{\alpha\beta} b - \tfrac{1}{2}(g_{\alpha}{}^{\alpha} - k_{\alpha\beta} X^{\alpha\beta}) b,$$

$$\delta\, f_{\gamma} = -\delta\, X^{\alpha\beta}\partial_{\alpha\beta} f_{\gamma} - \tfrac{1}{2}(g_{\alpha}{}^{\alpha} - k_{\alpha\beta} X^{\alpha\beta}) f_{\gamma} - (g_{\gamma}{}^{\alpha} - k_{\gamma\beta} X^{\beta\alpha}) f_{\alpha}$$

conformal weights of the fields

Sp(8) generators:

$$P_{\alpha\beta} = \frac{\partial}{\partial X^{\alpha\beta}}, \qquad L_{\alpha}{}^{\beta} = 2X^{\beta\gamma}\frac{\partial}{\partial X^{\gamma\alpha}}, \qquad K^{\alpha\beta} = X^{\alpha\gamma} X^{\beta\delta}\frac{\partial}{\partial X^{\gamma\delta}}$$

generators of $GL(4)$

$$[P,P]=0, \quad [K,K]=0, \quad [P,K]=L,$$
$$[L,L]=L, \quad [L,P]=P, \quad [L,K]=K$$

Hyperspace is a coset space: $P = \dfrac{Sp(8)}{GL(4)\times\rangle\, K}$

Hyperspace extension of $AdS(4)$

(*Bandos, Lukierski, Preitschopf, D.S. '99; Vasiliev '01*)

- 10d group manifold $Sp(4,R) \sim SO(2,3)$

$$AdS_4 = \frac{Sp(4)}{SO(1,3)} \qquad Sp(4) = \frac{Sp(8)}{GL(4)\times\rangle K} = P + K$$ - different $Sp(8)$ coset

Like Minkowski and $AdS(4)$ spaces, which are conformally flat, the flat hyperspace and $Sp(4)$ are (locally) related to each other by a "generalized conformal" transformation

<u>$Sp(2M)$ group manifolds are GL-flat</u> (*Plyushchay, D.S. & Tsulaia '03*)

Algebra of covariant derivatives on $Sp(4)$:

$$[\nabla_{\alpha\beta}, \nabla_{\gamma\delta}] = \tfrac{1}{2r}(C_{\alpha(\gamma}\nabla_{\delta)\beta} + C_{\beta(\gamma}\nabla_{\delta)\alpha}), \quad C_{\alpha\beta} = -C_{\beta\alpha},$$ r - is $Sp(4)$ (or AdS4) radius

$$\nabla_{\alpha\beta} = G_\alpha{}^\gamma G_\beta{}^\delta \frac{\partial}{\partial X^{\alpha\beta}}, \qquad G_\alpha{}^\gamma(X) = \delta_\alpha{}^\gamma + \tfrac{1}{4r} X_\alpha{}^\gamma; \qquad \Omega^{\alpha\beta}(X) = G_\gamma^{-1\alpha} G_\delta^{-1\beta} dX^{\gamma\delta}$$ $Sp(4)$ Cartan form

GL-flatness is important for the relation between the field equations in flat and $Sp(4)$ hyperspace

HS field equations in Sp(4)

(*Didenko and Vasiliev '03*; *Plyushchay, D.S. & Tsulaia '03*)

Flat hyperspace equations:

$$(\partial_{\alpha\beta}\partial_{\gamma\delta} - \partial_{\alpha\gamma}\partial_{\beta\delta})\, b(X) = 0$$

$$\partial_{\alpha\beta} f_\gamma(X) - \partial_{\alpha\gamma} f_\beta(X) = 0$$

$Sp(4)$ field equations (*Plyushchay, D.S. & Tsulaia '03*) :

Fermi: $$\nabla_{\alpha\beta}F_\gamma - \nabla_{\alpha\gamma}F_\beta = \tfrac{1}{4r}(C_{\beta(\alpha}F_{\gamma)} - C_{\gamma(\alpha}F_{\beta)})$$

Bose: $$\nabla_{\alpha\beta}\nabla_{\gamma\delta}B - \nabla_{\alpha\gamma}\nabla_{\beta\delta}B = \tfrac{1}{8r}(C_{\gamma(\delta}\nabla_{\alpha)\beta} - C_{\beta(\delta}\nabla_{\gamma)\alpha} - C_{\beta(\gamma}\nabla_{\alpha)\delta})B + \tfrac{1}{32r^2}(C_{\alpha(\beta}C_{\delta)\gamma} - C_{\alpha(\gamma}C_{\delta)\beta})B$$

Generalized conformal relations between flat and Sp(4) hyperfields

(Florakis, *D.S. & Tsulaia '14*)

$$B(X) = \sqrt{\det G}\, b(X), \qquad F_\alpha(X) = \sqrt{\det G}\, G_\alpha{}^\beta f_\beta(X), \qquad G_\alpha{}^\beta = \delta_\alpha{}^\beta + \tfrac{1}{4r} X_\alpha{}^\beta$$

$Sp(8)$ invariant correlation functions

In flat hyperspace (*Vasiliev '01, Vasiliev & Zaikin '03*) (*also Didenko & Skovrtsov '12*)

conformal weight

$$\langle b(X_1)\, b(X_2)\rangle = c_{bb}(\det|X_1 - X_2|)^{-\frac{1}{2}}$$

$$\langle f_\alpha(X_1)\; f_\beta(X_2)\rangle = c_{ff}\,(X_1 - X_2)^{-1}_{\alpha\beta}(\det|X_1 - X_2|)^{-\frac{1}{2}}$$

$$\langle b(X_1)\, b(X_2)\, b(X_3)\rangle = c_{bbb}(\det|X_1 - X_3|)^{-\frac{1}{4}}(\det|X_2 - X_3|)^{-\frac{1}{4}}(\det|X_1 - X_2|)^{-\frac{1}{4}}$$

In $Sp(4)$ hyperspace (*Florakis, D.S. & Tsulaia '14*)

$$\langle B(X_1)\, B(X_2)\rangle_{Sp(4)} = \sqrt{\det G(X_1)}\,\sqrt{\det G(X_2)}\,\langle b(X_1)\, b(X_2)\rangle_{flat}$$

$$\langle F_\alpha(X_1)\, F_\beta(X_2)\rangle_{Sp(4)} = \sqrt{\det G(X_1)}\,\sqrt{\det G(X_2)}\, G_\alpha{}^\gamma(X_1)\, G_\beta{}^\delta(X_2)\langle f_\gamma(X_1)\; f_\delta(X_2)\rangle_{flat}$$

$$\langle B(X_1)\, B(X_2)\, B(X_3)\rangle_{Sp(4)} = \sqrt{\det G(X_1)}\,\sqrt{\det G(X_2)}\,\sqrt{\det G(X_3)}\langle b(X_1)\, b(X_2)\, b(X_3)\rangle_{flat}$$

$$G_\alpha{}^\beta(X) = \delta_\alpha{}^\beta + \tfrac{1}{4r} X_\alpha{}^\beta$$

Four-point functions

Bosonic:

$$\Phi(X_1, X_2, X_3, X_4) = c_4 \prod_{ij,i<j} \frac{1}{(\det|X_{ij}|)^{\Gamma_{ij}}} \tilde{\Phi}\left(z, z'\right) ,$$

where z, z' are the two independent cross-ratios

$$X_{ij}^{\alpha\beta} = X_i^{\alpha\beta} - X_j^{\alpha\beta}$$

$$z = \det\left(\frac{|X_{12}||X_{34}|}{|X_{13}||X_{24}|}\right) , \quad z' = \det\left(\frac{|X_{12}||X_{34}|}{|X_{23}||X_{14}|}\right) .$$

Crossing symmetry then implies the constraint

$$\tilde{\Phi}(z, z') = \tilde{\Phi}\left(\frac{1}{z}, \frac{z'}{z}\right) = \tilde{\Phi}\left(\frac{z}{z'}, \frac{1}{z'}\right) .$$

Fermionic:

$$\langle F_\mu(X_1)F_\nu(X_2)F_\rho(X_3)F_\sigma(X_4)\rangle_{flat} = \prod_{i<j} \det|X_{ij}|^{-\frac{1}{3}} \left[(X_{12})^{-1}_{\mu\nu}(X_{34})^{-1}_{\rho\sigma}\Phi_{12,34}(z,z') \right.$$
$$\left. -(X_{13})^{-1}_{\mu\rho}(X_{24})^{-1}_{\nu\sigma}\Phi_{13,24}(z,z') + (X_{14})^{-1}_{\mu\sigma}(X_{23})^{-1}_{\nu\rho}\Phi_{14,23}(z,z')\right] .$$

As before, the functions $\Phi_{ij,k\ell}(z, z')$ are indeterminate functions of the crossing ratic
strained by crossing symmetry to satisfy

$$\Phi_{12,34}(z, z') = \Phi_{13,24}\left(\frac{1}{z}, \frac{z'}{z}\right) = \Phi_{14,23}\left(\frac{z}{z'}, \frac{1}{z'}\right) .$$

Supersymmetry in hyperspace

$$\Phi(X,\theta) = b(X) + f_\alpha(X)\theta^\alpha + \text{auxiliary fields}$$

$$D_\alpha = \tfrac{\partial}{\partial\theta^\alpha} + i\theta^\beta\partial_{\beta\alpha}, \qquad \{D_\alpha, D_\beta\} = 2i\partial_{\alpha\beta} = 2P_m\gamma^m_{\alpha\beta} + Z_{mn}\gamma^{mn}_{\alpha\beta}$$

$$[D_\alpha, D_\beta]\Phi(X,\theta) = 0 \quad - \quad OSp(1|8)\text{-invariant equations of motion}$$

Higher spin fields form an infinite dimensionsal D=4 supermultiplet

Osp(1|8) – invariant correlation functions of scalar superfields

$$\left\langle \Phi(X_1,\theta_1)\Phi(X_2,\theta_2)\right\rangle = c_2(|\det Z_{12}|)^{-\Delta}$$

$$\left\langle \Phi^{\Delta_1}(X_1,\theta_1)\Phi^{\Delta_2}(X_2,\theta_2)\Phi^{\Delta_3}(X_3,\theta_3)\right\rangle = c_3(|\det Z_{12}|)^{-k_1}(|\det Z_{23}|)^{-k_2}(|\det Z_{31}|)^{-k_3}$$

$$Z^{\alpha\beta}_{ij} = X^{\alpha\beta}_i - X^{\alpha\beta}_j - \tfrac{i}{2}\theta^\alpha_i\theta^\beta_j - \tfrac{i}{2}\theta^\beta_i\theta^\alpha_j$$

Conclusion

- Free theory of the infinite number of massless HS fields in 4d flat and AdS4 space has generalized conformal Sp(8) symmetry and can be compactly formulated in 10d hyperspace with the use of one scalar and one spinor field.
- Higher dimensional extension to $Sp(2M)$ invariant hyperspaces is straightforward *(Bandos, Lukierski, D.S. '99, Vasiliev '01,… …)*

  known *physically relevant cases* are

  M=2 (d=3), M=4 (d=4), M=8 (d=6), M=16 (d=10)

  describe conformal HS fields in corresponding space-times.
- M=32, d=11 - M-theory and E(11) *(P. West '07)*
- Is there any relation to Doubled Field Theory?
- Supersymmetric generalizations are available

  (Bandos et. al, Vasiliev et. al., Ivanov et. al., P. West, Florakis et. al…)
- Main problem: Whether one can construct an interacting field theory in hyperspace which would describe HS interactions in conventional space-time
  - Attempt via hyperspace SUGRA *(Bandos, Pasti, D.S., Tonin '04)*
- Hyperspace formualtion of HS fields was extended to incorporate one-form gauge connections [in unfolded setting] *(Vasiliev '07)*