Holographic Aspects of Large-N Vector Models

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Research co-financed by Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF), under the grants scheme "ARISTEIA II".

Outline

1. Motivations
2. The bosonic $O(N)$ vector model as a CFT
   - Setup
   - The free field theory
   - The skeleton graphs
3. Summary and outlook

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1. Test Holography and AdS/CFT beyond string theory.

- The $O(N)$ vector model: $O(1/N)$ anomalous dimensions of the $O(N)$-singlet higher-spin currents are all determined by γ_ϕ [W. Rühl - private communication]:

$$J_{(s)} \sim \phi^a \partial_{\{\mu_1} ...\partial_{\mu_s\}} \phi^a \,, \quad a = 1, 2, .., N$$

$$\Delta_s = s + 1 + 4\gamma_\phi \frac{s-2}{2s-1} + \cdots \,, \; s = 2k \,, \; k = 1, 2, .. \,, \; \gamma_\phi \sim O(1/N)$$

$$s \to \infty \,, \quad \Delta_s - s \approx 2\left(\frac{1}{2} + \gamma_\phi\right)$$

- Contrast with $\mathcal{N} = 4$ SYM: 1) No $\ln s$ growth that would signal the presence of gauge fields. 2) Hard to arise from rotating strings in AdS. (However, fast rotating ultrashort strings (particles?) in an AdS$_4$ black hole yield the T-independent result [Armoni, Barbon and A.C.P. (02)])

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Motivations

The conjectures

The $O(N)$ singlet sector of the bosonic vector model is dual to the simplest Vasiliev Theory on AdS_4 [Klebanov and Polyakov (02)].

An analogous conjecture for the $O(N)$ fermionic vector model - slightly complicated due to parity issues - [Leigh and A. C. P. , Sezgin and Sundell (03)]

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2. Compare the bulk and boundary OPE studies: understand how the bulk "emerges" from the boundary and vice versa.

- Diagrammatic $1/N$ "skeleton" expansion elucidates the OPE structure of the boundary theories and gives interesting results.
  However, extension of such techniques to the bulk is rather mysterious (see however [X. Bekaert et. al (14)]).

Sample further questions (still impenetrable in $d \geq 3$)

3. HS black-holes [S. Didenko et. al (09)] and boundary thermalization: The bosonic model realises the Mermin-Wagner theorem: $O(N)$ symmetry does not break for $T > 0$ - parity does break for $T > 0$. How is this realised in terms of HSs?
4. $N \to 0$ limit relevant to polymers?

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Questions: global symmetries in the boundary

- Vector models exhibit global and discrete symmetry breaking: the bosonic model $O(N) \to O(N-1)$, the fermionic model parity breaking.
- If there is holography without strings and branes, a) what is the bulk counterpart of the global $O(N)$ boundary symmetry and b) what is its breaking pattern?

Answer: singletons

[R. G. Leigh and A. C. P. (12)] (see also [O. Gelfond and M. Vasiliev (13)])

- At least to leading order in $1/N$, the bulk HS theory can be deformed by "eating" (i.e. integrating-in) singletons → these are pushed towards the boundary and induce the $N \to N+1$ shift.
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The relevant 2- and 3-pt functions $(x_{ij} = x_i - x_j)$

- $$\begin{aligned} \langle \phi^a(x_1)\phi^b(x_2)\rangle &= \frac{1}{x_{12}^{2\Delta_\phi}}\delta^{ab}\,, \quad \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{1}{x_{12}^{2\Delta}} \\ \langle \phi^a(x_1)\phi^b(x_2)\mathcal{O}(x_3)\rangle &= g_{\phi\phi\mathcal{O}}\frac{1}{(x_{12}^2)^{\Delta_\phi-\frac{1}{2}\Delta}(x_{13}^2x_{23}^2)^{\frac{1}{2}\Delta}}\delta^{ab} \end{aligned}$$
- The question: can conformal invariance alone determine the values of Δ_ϕ, Δ and $g_{\phi\phi\mathcal{O}}$?
- Older answer ("old bootstrap"): use an improved Dyson-Schwinger expansion for the above 2- and 3-pt functions that leads to algebraic equations for the critical parameters. [The St. Petersburg group (80-82)] , e.g. Δ_ϕ is known up to $O(1/N^3)$.

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- New bootstrap: focus on 4-pt functions and use the analytic properties of the conformal OPE (Newest bootstrap: use algorithmic techniques and guesswork to obtain numerical results [S. Rychkov et. al. (10)])

$$\begin{aligned}
\langle\phi^a(x_1)\phi^b(x_2)\phi^c(x_3)\phi^d(x_4)\rangle &\equiv \Phi^{abcd}(x_1,x_2,x_3,x_4)\\
&= \delta^{ab}\delta^{cd}\Phi_S(x_1,x_2,x_3,x_4)\\
&+ \mathcal{E}^{[ab,cd]}\Phi_A(x_1,x_2,x_3,x_4)\\
&+ \mathcal{T}^{(ab,cd)}\Phi_{st}(x_1,x_2,x_3,x_4)
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- These will be functions of v and Y related to the usual conformal ratios as

$$u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\,,\quad v=\frac{x_{12}^2x_{34}^2}{x_{14}^2x_{23}^2}\,,\; Y=1-\frac{v}{u}$$

with $v,Y\to 0$ as $x_{12}^2,x_{34}^2\to 0$.

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- We would need the following OPE

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\begin{aligned}
\phi^a(x_1)\phi^b(x_2) &= \sum_{\Delta_s} \frac{\delta^{ab}}{(x_{12}^2)^{\Delta_\phi - \frac{1}{2}\Delta_s}} \left[1 + \frac{g_{\phi\phi\mathcal{O}_s}}{C_{\mathcal{O}_s}}[\mathcal{O}_s(x_2)]\right], \\
&+ \sum_{\Delta_s'} \frac{\mathcal{E}^{[ab,cd]}}{(x_{12}^2)^{\Delta_\phi - \frac{1}{2}\Delta_s'}} \frac{g_{\phi\phi\mathcal{O}_s^{[cd]}}}{C_{\mathcal{O}_s^{[cd]}}}[\mathcal{O}_s^{[cd]}(x_2)] \\
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\end{aligned}
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- The $[\mathcal{O}_s]$'s represent the full contributions (i.e. including descendants). The $C_{\mathcal{O}_s}$'s are the 2-pt function normalisation constants and the $g_{\phi\phi\mathcal{O}_s}$'s are the corresponding 3-pt function couplings. We normalized to one the 2-pt function of the ϕ^a's.
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- We would also need

$$\phi^a(x_1)\mathcal{O}(x_2) = \frac{1}{(x_{12}^2)^{\frac{\Delta_\phi+\Delta}{2}}}\left[\frac{g_{\phi\phi\mathcal{O}}}{(x_{12}^2)^{-\frac{\Delta_\phi}{2}}}[\phi^a(x_2)] + \frac{g_{\phi\mathcal{O}F}}{C_F}\frac{[F^a(x_2)]}{(x_{12}^2)^{-\frac{\Delta_F}{2}}} + ..\right]$$

$$\mathcal{O}(x_1)\mathcal{O}(x_2) = \frac{1}{x_{12}^{2\Delta}}\left[C_{\mathcal{O}} + \frac{g_{\mathcal{O}}}{C_{\mathcal{O}}}\frac{[\mathcal{O}(x_2)]}{(x_{12}^2)^{-\frac{\Delta}{2}}} + \frac{g_{\mathcal{O}\mathcal{O}T}}{C_T}\frac{C_{\mu\nu}[T_{\mu\nu}(x_2)]}{(x_{12}^2)^{-\frac{d}{2}}} + ..\right]$$

- Inserting the OPEs into the 4-pt functions we obtain formulae like

$$\Phi(v,Y)=\sum_{\Delta_s}\frac{1}{(x_{12}^2x_{34}^2)^{\Delta_\phi}}\frac{g^2_{\phi\phi\mathcal{O}_s}}{C_{\mathcal{O}_s}}\mathcal{H}_{\Delta_s}(v,Y)$$

with $\mathcal{H}_{\Delta_s}(v,Y)$ the conformal partial wave (CPW) representing the contribution of the operator $\mathcal{O}_s$ and all its descendants into the 4-pt function.

- The CPW's are given is terms of a double series of the form

$$\mathcal{H}_{\Delta_s}(v,Y)=v^{\frac{1}{2}(\Delta_s-s)}\sum_{n,m=0}^{\infty}A_{nm}v^nY^m$$

- When the operators are conserved spin-s currents with $\Delta_s=d-2+s$ the leading singular term has the form

$$\mathcal{H}_{\Delta_s}(v,Y)=A_{0s}v^{\frac{1}{2}d-1}Y^s[1+O(v)]\cdots$$

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- Assuming the presence of one only scalar operator $\mathcal{O}$ with dimension $\Delta < d$ in the OPE, we have for the first few most singular terms

$$\Phi_S(v,Y) = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}}\left[1 + \frac{g_{\phi\phi\mathcal{O}}^2}{C_{\mathcal{O}}} v^{\frac{\Delta}{2}} {}_2F_1(\frac{\Delta}{2},\frac{\Delta}{2};\Delta;Y) + \frac{g_{\phi\phi T}^2}{4C_T} v^{\frac{d}{2}-1} Y^2 + \ldots\right]$$

- We need to match this with an explicit calculation. The obvious one is free field theory

$$\Phi_S(v,Y) = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}}\left[1 + v^{\Delta_\phi}\left(1 + \frac{1}{(1-Y)^{\Delta_\phi}}\right)\right]$$

- We first obtain that

$$\Delta_\phi = \frac{d}{2} - 1\,, \;\; \Delta = 2\Delta_\phi = d-2\,,$$

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$$\Delta_\phi = \frac{d}{2} - 1\,, \quad \Delta = 2\Delta_\phi = d-2\,,$$

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$$\mathcal{O}(x) = \frac{1}{\sqrt{2N}}\phi^a(x)\phi^a(x)\,, \;\Rightarrow C_{\mathcal{O}} = 1$$

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$$g^2_{\phi\phi\mathcal{O}} = \frac{2}{N}$$

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packages efficiently the contributions of an infinite number of even-spin HS currents, the normalization of their 2-pt functions and their 3-pt function couplings with the ϕ's. The latter are determined by HS Ward identities, hence the above expression "knows" about HS symmetry.

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- Going on, we find the OPE of Φ_A as

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that gives the leading contribution of a spin-1 conserved current J.

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- We notice he contribution of the even-spin HS currents.
- The disconnected graphs give the contribution of the form

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- For $s=0$ this corresponds to a scalar with $\Delta = 4\Delta_\phi$ i.e. $(\phi^2)^2$.
- For $s \neq 0$ these correspond to twist $t=2$ HS operators with $\Delta_s = d-2+s+t$. Notice that all higher-twist operators have been cancelled.

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- To deform the free theory we use an expansion in"skeleton graphs" built using just three ingredients: the (unit normalised) 2-pt functions of the operators $\phi^a(x)$, $O(x)$ (with dimension $\tilde{\Delta}$) and the 3-pt function

$$\langle \phi^a(x_1)\phi^b(x_2)O(x_3)\rangle = g_* \frac{1}{(x_{12}^2)^{\Delta_\phi - \frac{\tilde{\Delta}}{2}}(x_{13}^2 x_{24}^2)^{\frac{\tilde{\Delta}}{2}}}\delta^{ab}\,.$$

- The parameters $\tilde{\Delta}$ and g_*, as well as all other parameters (i.e. coupling and scaling dimensions) will be determined by studying the consistency of the skeleton expansion with the OPE.
- We also need to "amputate" using the inverse 2-pt functions

$$\begin{aligned}\delta^{ab}\Gamma(x_1,x_2,x) &\equiv \int d^d x_3 \langle \phi^a(x_1)\phi^b(x_2)O(x_3)\rangle\langle O(x_3)O(x)\rangle^{-1}\\ &= g_* \frac{f(\Delta_\phi,\tilde{\Delta},d)}{(x_{12}^2)^{\Delta_\phi-\frac{\tilde{\Delta}}{2}}(x_{13}^2x_{24}^2)^{\frac{d-\tilde{\Delta}}{2}}}\delta^{ab}\end{aligned}$$

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- This construction is an important simplification compared to the usual $1/N$ diagrammatic expansion of the vector model: the full vertices and 2-pt functions are used.
- The skeleton expansion for Φ_S will involve tree-exchange graphs with a single $O(x)$ internal line, ladder graphs with internal $O(x)$ and $\phi^a(x)$ lines etc...
- The leading exchange graph in the direct channel $x_{12}^2, x_{34}^2 \Rightarrow 0$ yields the remarkable formula

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This is remarkable since it corresponds to the CPWs of *both* the operator $O(x)$ but also its shadow operator with dimension $d - \tilde{\Delta}$.

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$$C(d,\tilde{\Delta}) = \frac{\Gamma(\tilde{\Delta})\Gamma(\tilde{\Delta}-\frac{d}{2})\Gamma^4(\frac{d}{2}-\frac{1}{2}\tilde{\Delta})}{\Gamma(d-\tilde{\Delta})\Gamma(\frac{d}{2}-\tilde{\Delta})\Gamma^4(\frac{d}{2})}$$

This is remarkable since it corresponds to the CPWs of *both* the operator $O(x)$ but also its shadow operator with dimension $d-\tilde{\Delta}$.

It can be shown that the presence of the shadow term is necessary for the graph to be analytic under a crossing transformation i.e. $x_2 \leftrightarrow x_3$.

- A given skeleton graph with $2n$ vertices has the shadow symmetry property

$$G(v,Y;\Delta) = [C(d,d-\Delta)]^n G(v,Y;d-\Delta)$$

- It is believed that the above property is related to the analyticity of the graph under crossing. Then, the full crossing symmetric 4-pt function can be obtained by adding to the direct channel the crossed terms.
- The crossed, box (and possibly all higher order) evaluate to the generic form

$$G(x_1,x_3,x_2,x_4) = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}} v^{\Delta_\phi} \sum_{n,m=0}^{\infty} \frac{v^n Y^m}{n!m!}[-a_{nm}\ln v + b_{nm}]$$

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Examples:

- Modifying the free result for $\Phi_S(v,Y)$ by the $\mathcal{O}$-exchange graph would imply the presence of three scalar operators with dimensions $< d$! To avoid that, we choose to cancel the free operator with $\Delta = 2\Delta_\phi$ with one of the two terms in the exchange graph. In fact, $C(d,\tilde{\Delta}) < 0$ for $\underline{2 < d < 6}$ and $\tilde{\Delta} < d$. This way we fix $g_*^2 \sim O(1/N)$ and also $d - \tilde{\Delta} = 2\Delta_\phi \Rightarrow \tilde{\Delta} = 2$.
- Modifying $\Phi_A(v,Y)$ we find

$$\begin{aligned}\Phi_A(v,Y) &= \Delta_\phi v^{\Delta_\phi} Y[1+..] + g_*^2 v^{\Delta_\phi} Y[-A_{00}\ln v + B_{00} + ..] \\ &= \frac{g_J^2}{C_J} v^{\frac{d}{2}-1} Y[1+..]\end{aligned}$$

- We need to kill the $\ln v$ terms in the first line, which is done if we assume that

$$\Delta_\phi = \frac{d}{2} - 1 + \frac{1}{N}\gamma_\phi\,, \;\Rightarrow \gamma_\phi = \frac{2\Gamma(d-2)}{\Gamma(\frac{d}{2}+1)\Gamma(\frac{d}{2})\Gamma(1-\frac{d}{2})\Gamma(\frac{d}{2}-2)}$$

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- For $\Phi_{\phi\mathcal{O}}(v,Y)$ we need exchange graphs involving the elementary scalar ϕ^a. These give *both* the CPW of ϕ^a but also of its shadow with $\Delta = 5/2$. One would also think that both contributions are $O(1/N)$. Quite remarkably, the latter contribution is singular, needs to be regularised and eventually gives rise to a $O(1)$ term in the 4-pt function.

$$C(d,\Delta_\phi) \to -\frac{N(\frac{d}{2}-2)^2}{\gamma_\phi \frac{d}{2}(\frac{d}{2}-1)} = -\frac{1}{g^2_{\phi\phi\mathcal{O}}} \;\Rightarrow\; g^2_{\phi\phi\mathcal{O}} C(d,\Delta_\phi) = -1 + O(1/N)$$

- This is necessary to correctly match with the OPE and make sure that the $\Delta = 5/2$ operator does not appear!
- This should be the boundary counterpart of the impossibility to quantize bulk singletons with standard Dirichlet boundary conditions: only the alternative quantization appears to give a positive measure.

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The issue with AdS graphs:

- A scalar field exchange graph in AdS in the direct channel gives

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  namely, the shadow contribution is missing.
- Nevertheless, one can show [L. Hoffmann, W. Ruhl and A. C. P. (00)] that such a graph is still analytic under a crossing transformation. This is due to some highly non-trivial Kummer-like relationships for ${}_3F_2$ functions!
- Also notice that there are no box and papillon graphs in the AdS description of $\Phi_{\mathcal{O}}$. Hence, it is necessary to consider the bulk vertices of all HS gauge fields. This was not necessary in the field theory description, hence the single $\phi\phi\mathcal{O}$ vertex was sufficient.

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- It is a formidable task to compare boundary skeleton expansion and the bulk Witten graphs, although they arguably describe the same theory.
  Hint:
  Boundary skeleton graphs do not have HS exchanges: I can built a HS theory using a single scalar vertex. But they have shadow-symmetry properties, and this is the part "speaking" to HS coming from the free theory.
  Bulk graphs do not have shadow-symmetry: but to built the theory one would need all HS exchanges. Namely, they include the "free part" that was actually "cancelled" by the shadow term in the skeleton graphs.