Divergences in Maximal SYM Theories in Diverse Dimensions

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Based on: JHEP 1311 (2013) 065, arXiv:1308.0117 [hep-th]
JHEP 1404 (2014) 121, arXiv:1402.1024 [hep-th]
Phys.Lett. B 734 (2014) 111, arXiv:1404.6998 [hep-th]
JHEP (2015), arXiv:1508.05570 [hep-th]

Motivation

Maximal SYM

D=4 N=4
D=6 N=2
D=8 N=1
D=10 N=1

- **Partial or total cancellation of UV divergences (all bubble and triangle diagrams cancel)**
- **First UV divergent diagrams at D=4+6/L**
- **Conformal or dual conformal symmetry**
- **Common structure of the integrands**

Bern, Dixon &Co 10
Drummond, Henn, Korchemsky, Sokatchev 10
Arkani-Hammed 12

Object: Helicity Amplitudes on mass shell with arbitrary number of legs and loops (S-matrix)

The case: Planar limit $N_c \to \infty,\ g^2_{YM} \to 0$ and $g^2_{YM} N_c$ - fixed

The aim: to get all loop (exact) result

UV & IR Divergences

D=4 N=4

- **No UV divergences in all loops**
- **IR & Collinear Divs on shell**

BDS conjecture *Bern, Dixon, Smirnov 05*

$$\mathcal{M}_n \equiv \frac{A_n}{A_n^{tree}} = 1 + \sum_{L=1}^{\infty} \left(\frac{g^2 N_c}{16\pi^2}\right)^L M_n^{(L)}(\epsilon) = \exp\left[\sum_{l=1}^{\infty} \left(\frac{g^2 N_c}{16\pi^2}\right)^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\Big)\right]$$

$$\mathcal{M}_n(\epsilon) = \exp\left[-\frac{1}{8}\sum_{l=1}^{\infty}\left(\frac{g^2 N_c}{16\pi^2}\right)^l \left(\frac{\gamma_{cusp}^{(l)}}{(l\epsilon)^2} + \frac{2G_0^{(l)}}{l\epsilon}\right)\sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{l\epsilon} + \frac{1}{4}\sum_{l=1}^{\infty}\left(\frac{g^2 N_c}{16\pi^2}\right)^l \gamma_{cusp}^{(l)} F_n^{(1)}(0) + C(g)\right]$$

IR & Collinear Divs in dimensional regularization **Cusp anom dim**

$$M_4^{(1-loop)}(\epsilon) = A_4^{(1-loop)}/A_4^{(tree)} = \frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\Big[\frac{1}{\epsilon^2}\left((\frac{\mu^2}{s})^{\epsilon} + (\frac{\mu^2}{-t})^{\epsilon}\right) - \frac{1}{2}\log^2\left(\frac{s}{-t}\right) - \frac{\pi^2}{3}\Big] + \mathcal{O}(\epsilon)$$

UV & IR Divergences

D=6 N=2

N=(1,1)

- **No IR & Collinear divergences in all loops**
- **UV Divs starting from L=6/(D-4)=3 loops**

D=8 N=1

- **No IR & Collinear divergences in all loops**
- **UV Divs starting from L=[6/(D-4)]=1 loops**

D=10 N=1

- **No IR & Collinear divergences in all loops**
- **UV Divs starting from L=6/(D-4)=1 loops**

Compactification on a torus of higher dim maximal SYM theories gives lower dimensional maximal SYM theories

Colour decomposition

Colour ordered amplitude

$$\mathcal{A}_n^{a_1\dots a_n}(p_1^{\lambda_1}\dots p_n^{\lambda_n}) = \sum_{\sigma\in S_n/Z_n} Tr[\sigma(T^{a_1}\dots T^{a_n})]A_n(\sigma(p_1^{\lambda_1}\dots p_n^{\lambda_n})) + \mathcal{O}(1/N_c)$$

Planar Limit $N_c \to \infty,\ g_{YM}^2 \to 0$ and $g_{YM}^2 N_c$ - fixed **This is what we calculate**

Four-point amplitude

$$A_4^{(1),\text{phys.}}(1,2,3,4) = T^1 A_4^{(0)}(1,2,3,4) M^{(1)}(s,t) + T^2 A_4^{(0)}(1,2,4,3) M^{(1)}(s,u) + T^3 A_4^{(0)}(1,4,2,3) M^{(1)}(t,u).$$

$$T^1 = \mathrm{Tr}(T^{a1}T^{a2}T^{a3}T^{a4}) + \mathrm{Tr}(T^{a1}T^{a4}T^{a3}T^{a2}),$$
$$T^2 = \mathrm{Tr}(T^{a1}T^{a2}T^{a4}T^{a3}) + \mathrm{Tr}(T^{a1}T^{a3}T^{a4}T^{a2}),$$
$$T^3 = \mathrm{Tr}(T^{a1}T^{a4}T^{a2}T^{a3}) + \mathrm{Tr}(T^{a1}T^{a3}T^{a2}T^{a4})$$

Tree level amplitude usually has a simple universal form proportional to the delta function (conservation of momenta), in SUSY case - conservation of supercharge in on shell momentum superspace

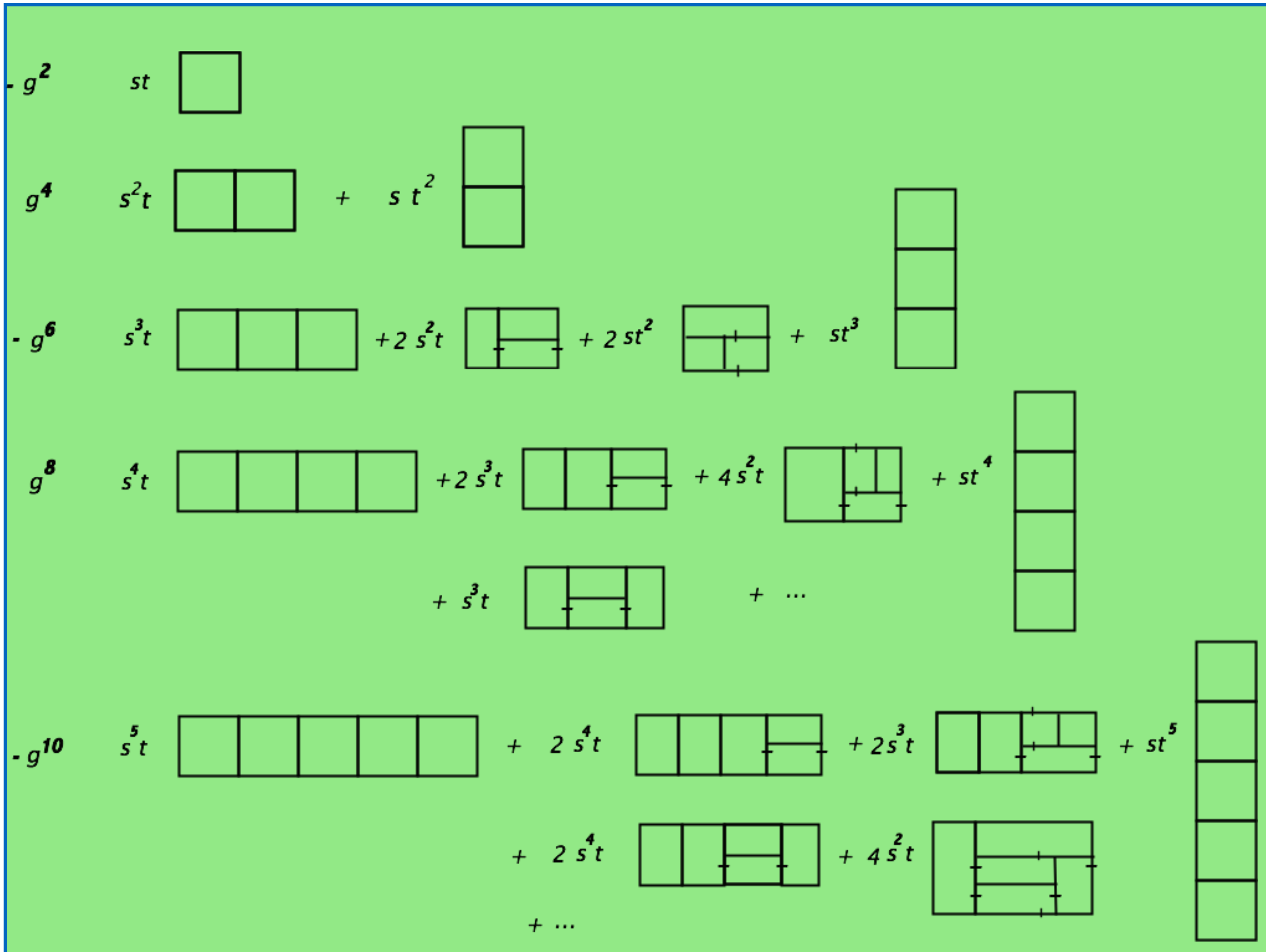
Perturbation Expansion for the Amplitudes for any D

A_4/A_4^{tree}

No bubbles
No Triangles

First UV div at L=[6/(D-4)] loops

IR finite



1
2
4
15
60

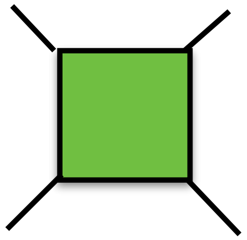
T. Dennen Yu-yin Huang 10 ,
S.Caron-Huot D.O'Connell 10

Universal expansion for any D in maximal SYM due to Dual conformal invariance

D=6 N=2

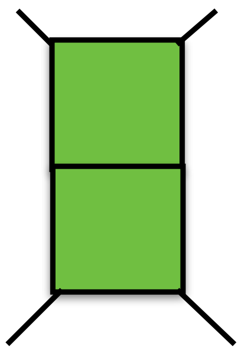
Perturbation Expansion for the Amplitudes

Exact calculation



$$p_i^2 = 0, \;\; m = 0$$

$$B_1(s,t) = \frac{\pi^3}{(2\pi)^6}\,\frac{b_2(x)}{s+t}, \quad b_2(x) = \frac{L^2(x)+\pi^2}{2}, \quad L(x) \doteq \log(x), \quad x = \frac{t}{s}$$



$$B_2(s,t) = \left(\frac{\pi^3}{(2\pi)^6}\right)^2 \left(\frac{b_4(x)}{t} + \frac{b_3(x)}{s+t}\right)$$

Anastasiou, Tausk, Tejeda-Yeomans , 00
Bork,Kazakov,Vlasenko, 13

$$\begin{aligned} b_4(x) &= \left(2\zeta_3 - 2Li_3(-x) - \frac{\pi^2}{3}L(x)\right)L(1+x) + \left(\frac{1}{2}L(x) + \frac{\pi^2}{2}\right)L^2(1+x) \\ &+ \left(2L(x)L(1+x) - \frac{\pi^2}{3}\right)Li_2(-x) + 2L(x)S_{1,2}(-x) - 2S_{2,2}(-x) \end{aligned}$$

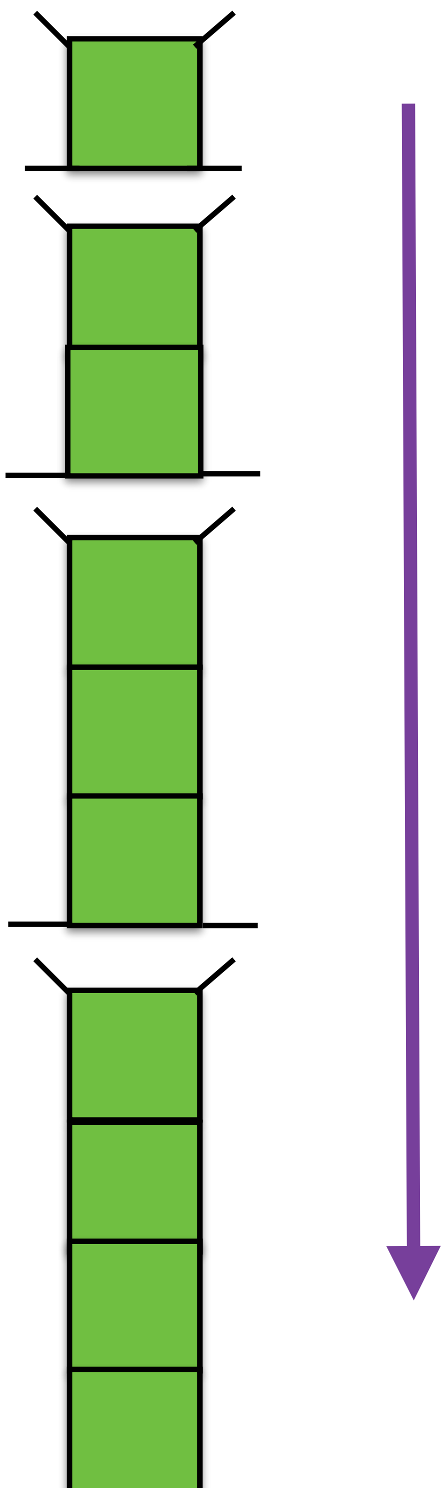
$$b_3(x) = -2\zeta_3 + \frac{\pi^2}{3}L(x) - \left(L(x)+\pi^2\right)L(1+x) - 2L(x)Li_2(-x) + 2Li_3(-x)$$

Regge Limit $\quad s \to \infty, \;\; t < 0, \;\; fixed$

$$B_1(s,t) \sim \frac{1}{2}L^2(x) \qquad\qquad B_2(s,t) \sim \frac{1}{12}L^4(x)$$

Perturbation Expansion for the Amplitudes
D=6 Example

Leading Logarithms



UV finite

Regge Limit $s \to \infty, \quad t < 0, \ fixed$

$$B_n(t,s) \simeq \frac{1}{s}\frac{L^{2n}(x)}{n!(n+1)!}, \qquad L \equiv \log(s/t)$$

Bork,Kazakov,Vlasenko, 13

$$\left.\frac{A_4}{A_4^{(0)}}\right|_{L.L.} = \sum_{n=0}^{\infty} \frac{(-g^2t/2)^n L^{2n}(x)}{n!(n+1)!}, \quad \text{where} \quad g^2 \equiv \frac{g_{YM}^2 N_c}{64\pi^3}.$$

$$\sum_{n=0}^{\infty} \frac{(-g^2t/2)^n L^{2n}(x)}{n!(n+1)!} = \frac{I_1(2y)}{y}, \qquad y \equiv \sqrt{g^2|t|/2}\ L(x)$$

$$\boxed{\left.\frac{A_4}{A_4^{(0)}}\right|_{L.L.} \sim \left(\frac{s}{t}\right)^{\alpha(t)-1}}$$

!

Regge behaviour

Exact for $N_c \to \infty$

$$\alpha(t) = 1 + 2\sqrt{g^2|t|/2} = 1 + \sqrt{\frac{g_{YM}^2 N_c |t|}{32\pi^3}}$$

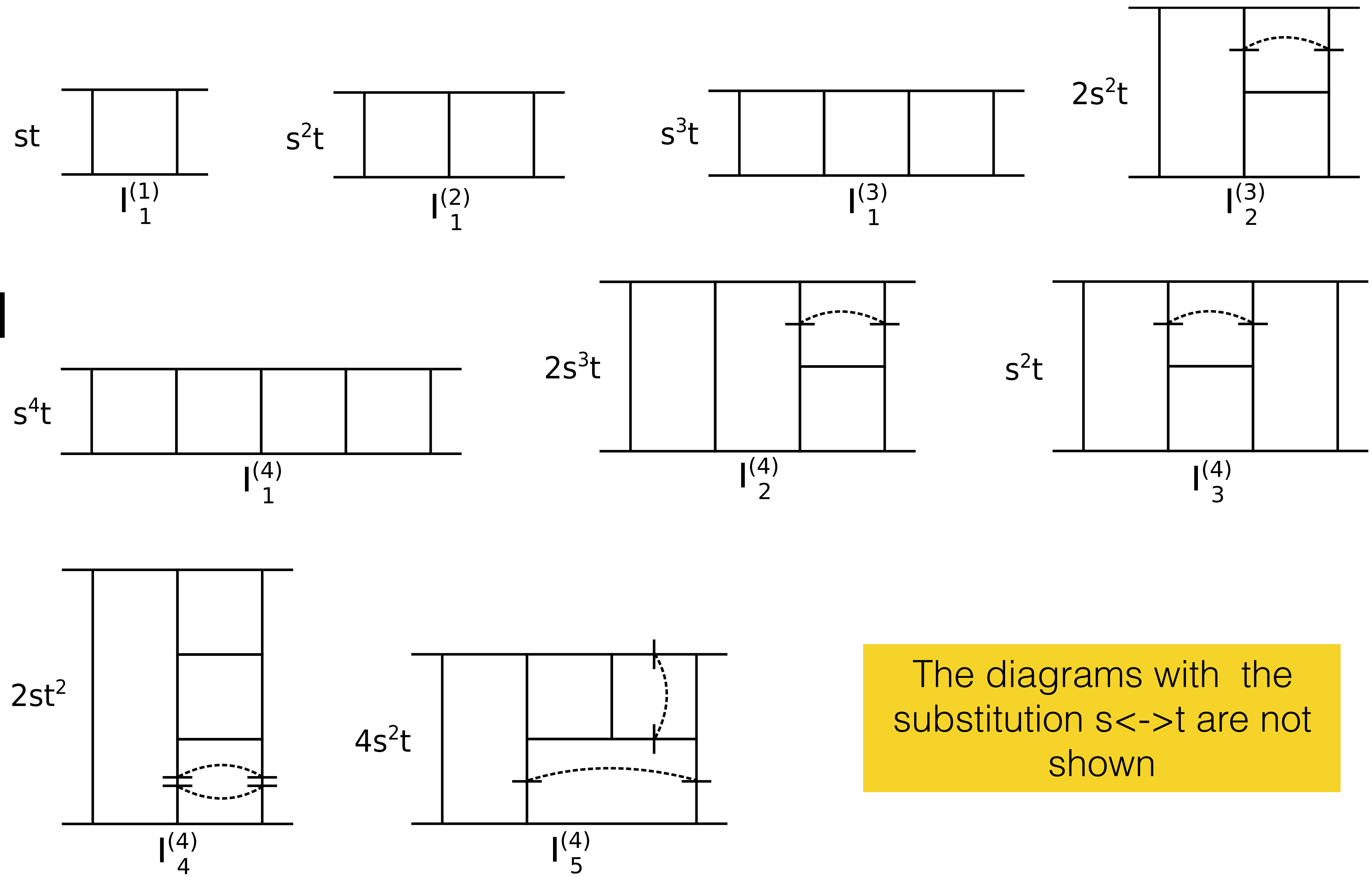
Perturbation Expansion for the Amplitudes

<u>Leading Divergences</u> The master integrals with leading divergences up to four loops

D=6 N=2

D=8 N=1

D=10 N=1



The diagrams with the substitution s<->t are not shown

Everything was checked also numerically!

Leading Divergences from Generalized «Renormalization Group»

- **In renormalizable theories the leading divergences can be found from the 1-loop term due to the renormalization group, in particular, for a single coupling theory the coefficient of $1/\epsilon^n$ in n loops in given by**

$$a_n^{(n)} = (a_1^{(1)})^n$$

- **In non-renormalizable theories the leading divergences can be also found from 1-loop due to locality and R-operation**

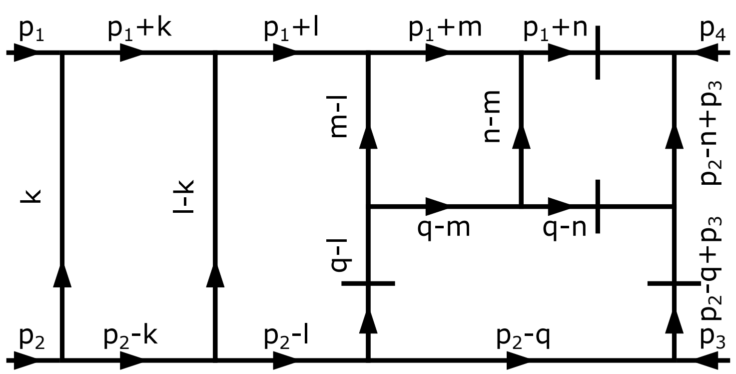
$$\mathcal{R}'G = 1 - \sum_{\gamma} K\mathcal{R}'_{\gamma} + \sum_{\gamma,\gamma'} K\mathcal{R}'_{\gamma} K\mathcal{R}'_{\gamma'} - ...,$$

$$\mathcal{R}'G_n = \frac{A_n(\mu^2)^{n\epsilon}}{\epsilon^n} + \frac{A_{n-1}(\mu^2)^{(n-1)\epsilon}}{\epsilon^n} + ... + \frac{A_1(\mu^2)^{\epsilon}}{\epsilon^n}$$

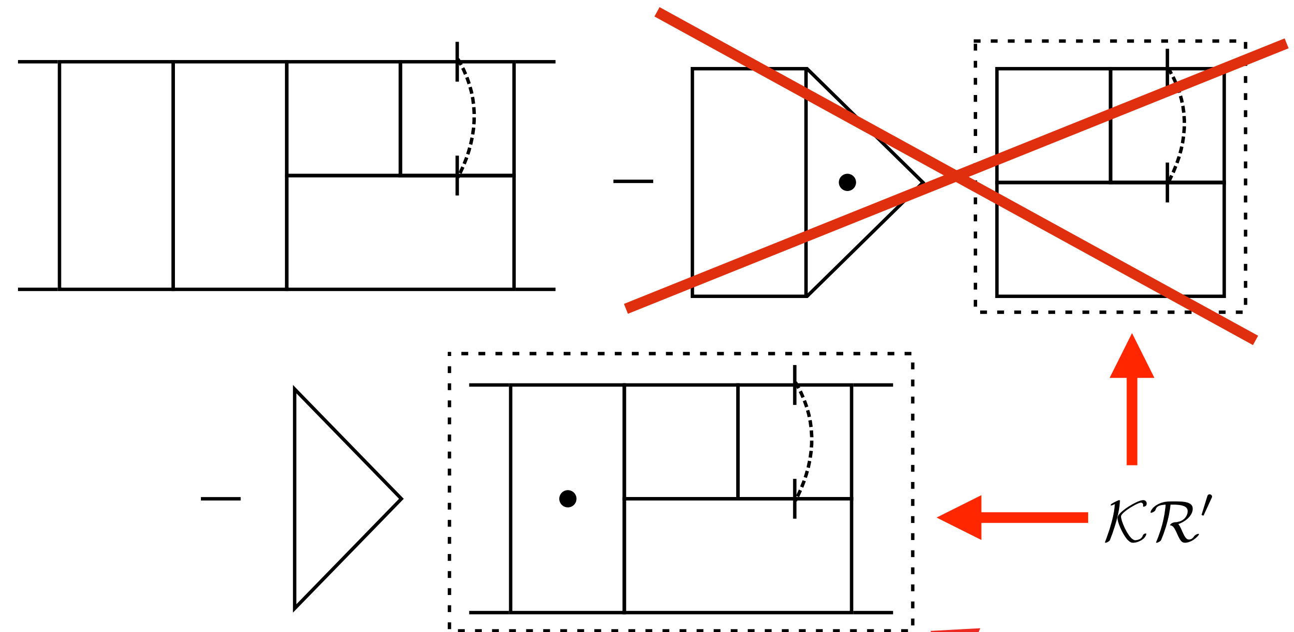
All terms like $(log\mu^2)^m/\epsilon^k$ **should cancel** $\rightarrow$ $\boxed{A_n = (-1)^{n-1}\frac{A_1}{n}}$

Leading pole **Coeff of 1 loop graph**

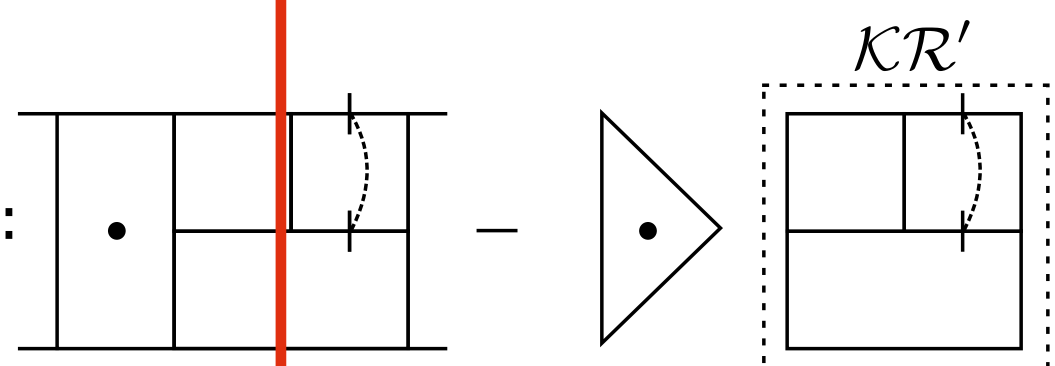
$\mathbf{R}'$- operation and Leading Divergences D=6 example



R':



R':



$$\mathcal{R}' : A_4 \frac{\mu^{4\epsilon}}{\epsilon^2} - \left(-\frac{1}{6\epsilon}\right)\left(-\frac{\mu^{\epsilon}}{6\epsilon} 2p_3(2p_2 - k + p_1)\right)$$

$$A_4 = \frac{2p_3(2p_2 - k + p_1)}{4 \cdot 36}$$

$$\mathcal{KR}' = \frac{2p_3(2p_2 - k + p_1)}{4 \cdot 36\epsilon^2}\mu^{4\epsilon} - \frac{2p_3(2p_2 - k + p_1)}{36\epsilon^2}\mu^{\epsilon} = -3\frac{2p_3(2p_2 - k + p_1)}{4 \cdot 36\epsilon^2}$$

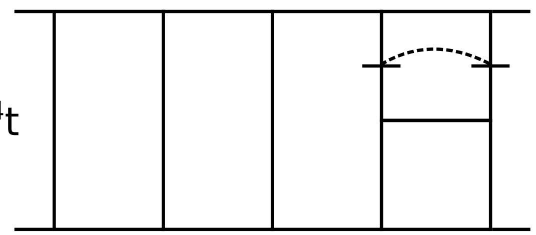
$$\boxed{L.P. = \frac{s - t/4}{30 \cdot 36 \cdot \epsilon^3}}$$

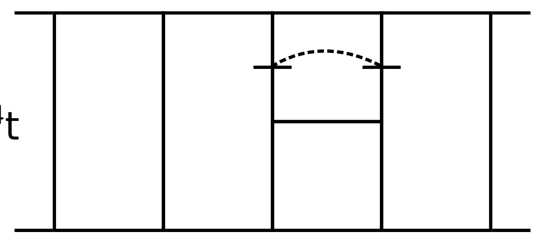
The leading Divergences from 1 to 4 loops

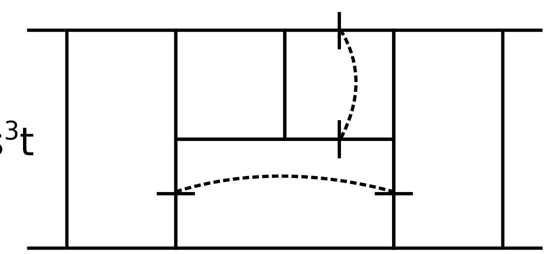
MI	Comb	$D=6$	$D=8$	$D=10$
$I_1^{(1)}$	st	conv	$\frac{1}{3!\epsilon}$	$\frac{s+t}{5!\epsilon}$
$I_1^{(2)}$	s^2t	conv	$-\frac{s}{3!4!\epsilon^2}$	$\frac{-s^2(8s+2t)}{5!7!\epsilon^2}$
$I_1^{(3)}$	s^3t	conv	$\frac{s^2}{4!5!\epsilon^3}$	$\frac{-2s^4(135s+11t)}{5!7!7!3\epsilon^3}$
$I_2^{(3)}$	$2s^2t$	$-\frac{1}{6\epsilon}$	$\frac{s(3s^2-2st+t^2)}{3!4!5!9\epsilon^3}$	$\frac{-s^2\left(14s^4-10s^3t+\frac{33}{5}s^2t^2-\frac{19}{5}st^3+\frac{8}{5}t^4\right)}{5!7!7!9\epsilon^3}$
$I_1^{(4)}$	s^4t	conv	$-\frac{210s^3}{3!4!5!6!\epsilon^4}$	$\frac{-32s^6(99s+2t)}{5!7!7!7!3\epsilon^4}$
$I_2^{(4)}$	$2s^3t$	$\frac{1}{48\epsilon^2}$	$\frac{s^2\left(-\frac{430}{21}s^2+\frac{4}{9}st-\frac{1}{18}t^2\right)}{3!4!5!6!\epsilon^4}$	$\frac{-2s^4\left(\frac{1502144}{33}s^4-\frac{1085791}{33}s^3t+\frac{2044}{5}s^2t^2-\frac{1001}{15}st^3+\frac{112}{15}t^4\right)}{5!7!7!7!7!\epsilon^4}$
$I_3^{(4)}$	s^3t	$\frac{1}{24\epsilon^2}$	$\frac{s^2\left(-\frac{20}{3}s^2+\frac{8}{9}st-\frac{1}{9}t^2\right)}{3!4!5!6!\epsilon^4}$	$\frac{-28s^4\left(8512s^4-1043s^3t+\frac{876}{5}s^2t^2- -\frac{143}{5}st^3+\frac{16}{5}t^4\right)}{5!7!7!7!7!3\epsilon^4}$
$I_4^{(4)}$	$2s^2t$	$\sim\frac{1}{\epsilon}$	$\frac{s\left(-\frac{45}{14}s^4+\frac{18}{7}s^3t-\frac{27}{14}s^2t^2+\frac{9}{7}st^3-\frac{9}{14}t^4\right)}{3!4!5!6!\epsilon^4}$	$\frac{-s^2\left(-\frac{7504}{1287}s^7+\frac{7819}{1716}s^6t-\frac{1475}{429}s^5t^2+\frac{12745}{5148}s^4t^3-\frac{716}{429}s^3t^4+\frac{1747}{1716}s^2t^5-\frac{673}{1287}st^6+\frac{105}{572}t^7\right)}{5!7!7!7!\epsilon^4}$
$I_5^{(4)}$	$4s^2t$	$\frac{t-s}{3\cdot 48\epsilon^2}$	$\frac{s\left(-\frac{15}{28}s^4+\frac{25}{63}s^3t-\frac{65}{252}s^2t^2+\frac{5}{42}st^3-\frac{1}{28}t^4\right)}{3!4!5!6!\epsilon^4}$	$\frac{-4s^2\left(-\frac{95200}{143}s^7+\frac{67634}{143}s^6t-\frac{225008}{715}s^5t^2+\frac{136514}{715}s^4t^3-\frac{6608}{65}s^3t^4+\frac{6706}{143}s^2t^5-\frac{7420}{429}st^6+\frac{1715}{429}t^7\right)}{5!7!7!7!7!\epsilon^4}$

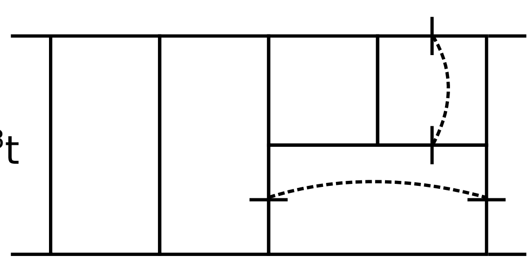
The leading Divergences from 5 loops

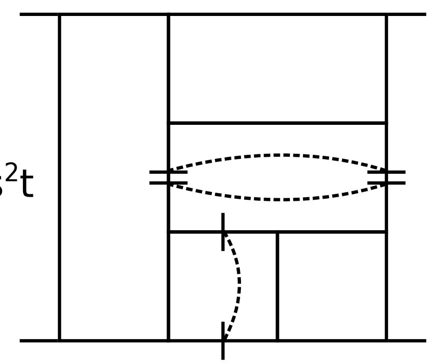
D=6 N=2 **Leading Divergences** 5 loops

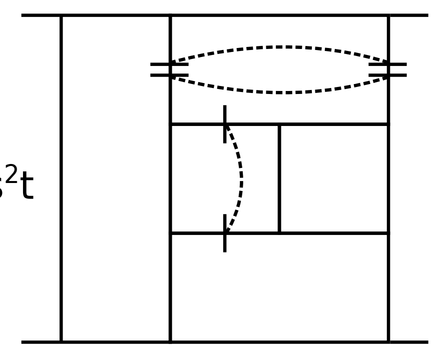
2s^{4}t

$I^{(5)}_1$

2s^{4}t

$I^{(5)}_2$

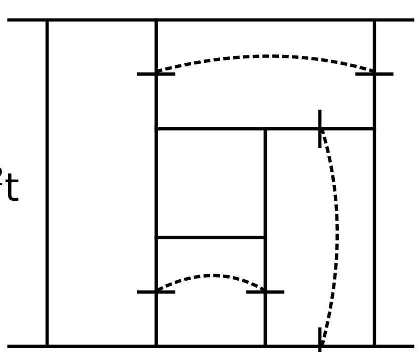
2s^{3}t

$I^{(5)}_3$

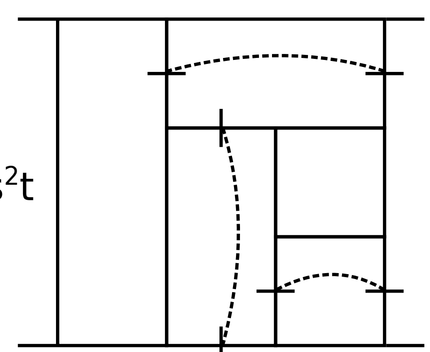
4s^{3}t

$I^{(5)}_4$

4s^{2}t

$I^{(5)}_5$

2s^{2}t

$I^{(5)}_6$

The diagrams with the substitution s<->t are not shown

4s^{2}t

$I^{(5)}_8$

4s^{2}t

$I^{(5)}_7$

MI	$I_1^{(5)}$	$I_2^{(5)}$	$I_3^{(5)}$	$I_4^{(5)}$
Comb	$2s^4t$	$2s^4t$	$4s^3t$	$2s^3t$
Int	$-\frac{1}{\epsilon^3}\frac{3}{36\cdot 40}$	$-\frac{1}{\epsilon^3}\frac{9}{36\cdot 40}$	$\frac{1}{\epsilon^3}\frac{s-t/4}{36\cdot 15}$	$\frac{1}{\epsilon^3}\frac{s-t/4}{36\cdot 30}$
MI	$I_5^{(5)}$	$I_6^{(5)}$	$I_7^{(5)}$	$I_8^{(5)}$
comb	$4s^2t$	$2s^2t$	$4s^2t$	$4s^2t$
Int	$-\frac{1}{\epsilon^3}\frac{s^2-st+t^2}{36\cdot 80}$	$-\frac{1}{\epsilon^3}\frac{s^2-st+t^2}{36\cdot 40}$	$\frac{1}{\epsilon^3}\frac{s^2-st+t^2/3}{36\cdot 80}$	$\frac{1}{\epsilon^3}\frac{s^2-st+t^2/3}{36\cdot 80}$

Numerical evaluation of Integrals

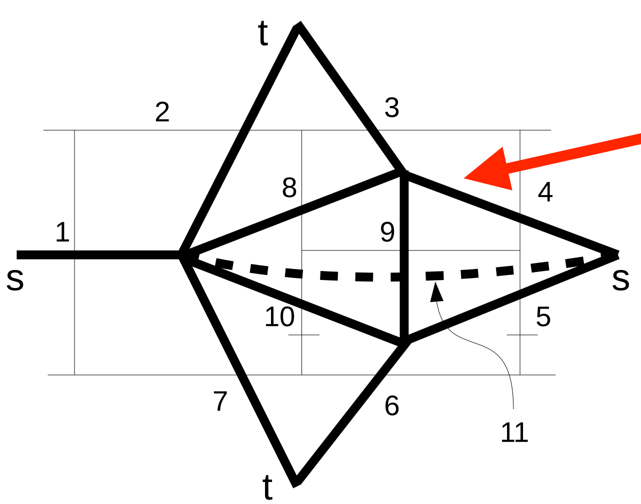
D=6 N=2 **Leading Divergences** α **-representation**

$$I(s,t,m_i) = \frac{(\pi)^{DL/2}}{\prod\limits_{i=1}^{n}\Gamma(\lambda_i)}\left(\left(\prod_{i=n+1}^{n+k}(-\partial_{\alpha_i})^{\kappa_i}\right)\int_0^\infty \frac{d\alpha_1 \ldots d\alpha_n}{U^{d/2}} e^{-V/U-\sum\limits_{j=1}^{n} m_j\alpha_j}\right)\Big|_{\alpha_{n+1}=\ldots=\alpha_{n+k}=0}$$

s=t=0, m $\neq 0$

$$\tilde{G}^{(D=8)}_{i,3-i}(s=0,t=0,m_i) = (\pi)^{3D/2}\left((-\partial_{\alpha_{11}})\int_0^\infty \frac{d\alpha_1 \ldots d\alpha_{10}(-P_s)^i(-P_t)^{3-i}}{U^{d/2+3}} e^{-\sum\limits_{j=1}^{10} m_j\alpha_j}\right)\Big|_{\alpha_{11}=0}$$

Numerator



Dual graph

graph	term	numerical	exact
$I_1^{(4)}$	$s^0\, t^0$	0	0
$I_2^{(4)}$	$s^0\, t^0$	$0.0416652(17)$	$1/24$
$I_3^{(4)}$	$s^0\, t^0$	$0.0208328(7)$	$1/48$

graph	term	numerical	exact
$I_1^{(4)}$	s^3	$-209.997(5)$	-210
$I_2^{(4)}$	s^4	-6.6661(10)	-20/3
	$s^3 t$	0.888900(24)	8/9
	$s^2 t^2$	-0.1111105(7)	-1/9
$I_3^{(4)}$	s^4	-20.4765(8)	-430/21
	$s^3 t$	0.444420(25)	4/9
	$s^2 t^2$	-0.0555541(10)	-1/18

Comparison with analytical evaluation

Perturbation Expansion for the Amplitudes

D=6 N=2

Result up to 5 loops

Leading Divergences

$$\boxed{L.P. = 2stg^4\left[g^2\frac{s+t}{6\epsilon} + g^4\frac{s^2+st+t^2}{36\epsilon^2} + g^6\frac{s^3+\frac{2}{5}s^2t+\frac{2}{5}st^2+t^3}{216\epsilon^3}\right]}$$

Geom progression !?

Leading powers of s > 0

$$\sum_{n=1}^{\infty}\left(\frac{g^2 s}{6\epsilon}\right)^n = \frac{\frac{g^2 s}{6\epsilon}}{1-\frac{g^2 s}{6\epsilon}}$$

Pole!

$\epsilon \to +0$

Leading powers of t < 0

$$\sum_{n=1}^{\infty}\left(\frac{g^2 t}{6\epsilon}\right)^n = \frac{\frac{g^2 t}{6\epsilon}}{1-\frac{g^2 t}{6\epsilon}}$$

-1

$\epsilon \to +0$

Compare D=4 YM

$$g^2 = \frac{g_B^2}{1-\frac{11C_2}{3}\frac{g_B^2}{\epsilon}}$$

General case will be given below

Perturbation Expansion for the Amplitudes

D=8 N=1 **Leading Divergences** **Result up to 4 loops**

$$L.P. = -st\left[g^2\frac{1}{3!\epsilon}+g^4\frac{s^2+t^2}{3!4!\epsilon^2}+g^6\frac{4}{3}\frac{15s^4-s^3t+s^2t^2-st^3+15t^4}{3!4!5!\epsilon^3}\right.$$
$$\left.+\ g^8\frac{1}{63}\frac{16770s^6-536s^5t+412s^4t^2-384s^3t^3+412s^2t^4-536st^5+16770t^6}{3!4!5!6!\epsilon^4}\right].$$

D=10 N=1 **Leading Divergences** **Result up to 4 loops**

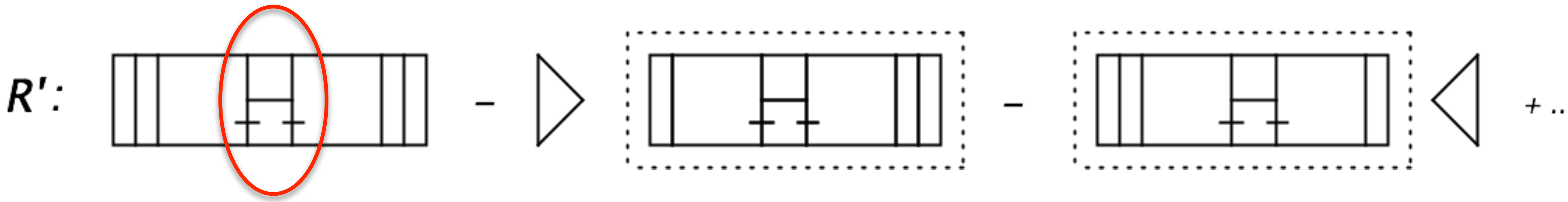
$$L.P. = -st\left[g^2\frac{s+t}{5!\epsilon}+g^4\frac{8s^4+2s^3t+2st^3+8t^4}{5!7!\epsilon^2}\right.$$
$$+\ g^6\frac{2(2095s^7+115s^6t+33s^5t^2-11s^4t^3-11s^3t^4+33s^2t^5+115st^6+2095t^7)}{5!7!7!45\epsilon^3}$$
$$+\ g^8\frac{32(211218880s^{10}+753490s^9t-1395096s^8t^2+1125763s^7t^3-916916s^6t^4}{13!7!7!5!5\epsilon^4}$$
$$\left.\frac{+843630s^5t^5-916916s^4t^6+1125763s^3t^7-1395096s^2t^8+753490st^9+211218880t^{10})}{13!7!7!5!5\epsilon^4}\right].$$

Doesn't look like Geom progression anymore, however, coefficients grow slowly

R-operation and Recurrence Relation

D=6 N=2

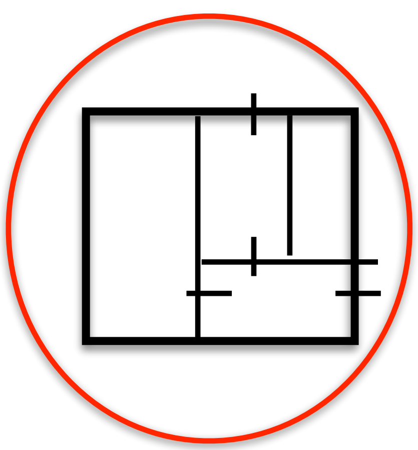
Horizontal boxes + tennis court



A_n $\qquad\qquad$ A_{n-1}

$$nA_n = -A_{n-1} \quad \longrightarrow \quad A_n = (-1)^n \frac{2}{n!} \qquad (-g^2 s)^n$$

Horizontal boxes + double tennis court



$$nA_n^t = -\frac{1}{3} A_{n-1}^t, \qquad nA_n^s = -A_{n-1}^s + \frac{1}{3} A_{n-1}^t$$

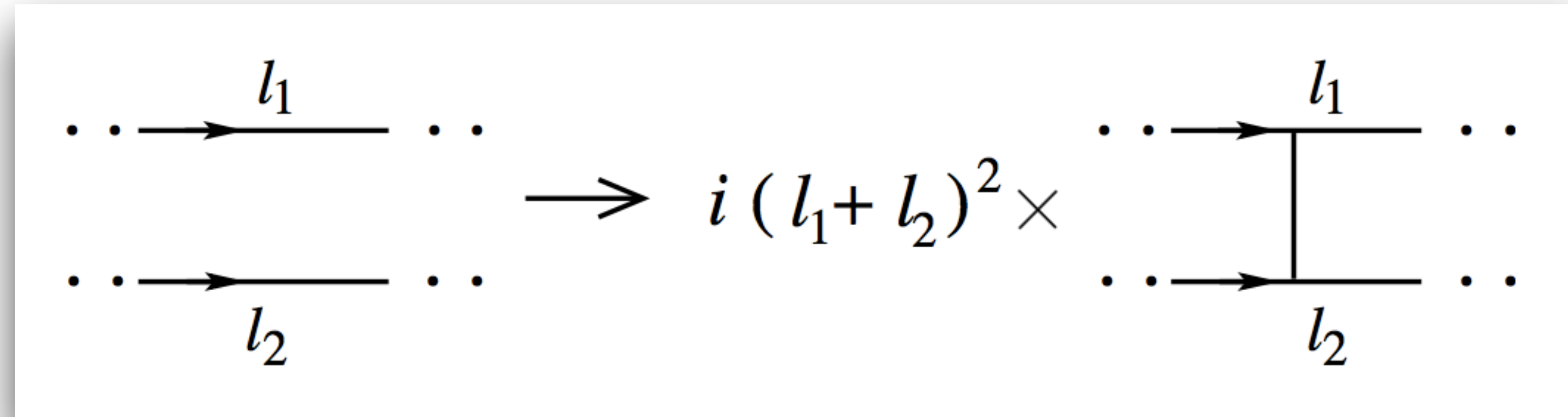
$$A_n^t = \frac{(-1)^n}{3^{n-3}} \frac{1}{n!}, \qquad A_n^s = \frac{1}{2} \frac{(-1)^n}{3^{n-3}} \frac{1}{n!} - \frac{1}{2} (-1)^n \frac{1}{n!}$$

$$(-g^2 s)^{n-1} (-g^2 t) \qquad\qquad (-g^2 s)^n$$

- **Similar relations one can get for all other series**
- **All of them have 1/n! behavior**
- **Number of these series group as n!**

All loop Exact Recurrence Relation

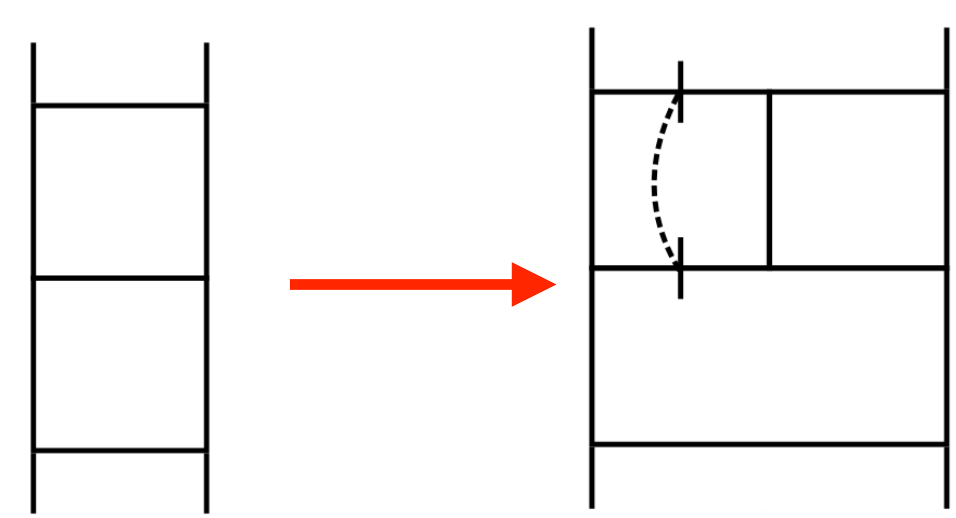
Some dual conformal integrals are generated by so called "Rung rule"



Not all dual conformal integrals are generated in this way, but integrals with <u>maximal number of divergent subgraphs</u> dose!

This allows one to write recursion relations for leading UV divergences!

2-loop -> 3-loop example



All loop Exact Recurrence Relation

D=6 N=2

s-channel term $S_n(s,t)$ **t-channel term** $T_n(s,t)$ $T_n(s,t)=S_n(t,s)$

Exact relation for ALL diagrams

$$\boxed{nS_n(s,t) = -2s\int_0^1 dx\int_0^x dy\ (S_{n-1}(s,t')+T_{n-1}(s,t'))}$$

$$n\geq 4$$

$$t' = t(x-y)-sy$$

$$S_3=-s/3,\ T_3=-t/3$$

Summation $$\Sigma_k(s,t,z)=\sum_{n=k}^{\infty}(-z)^n S_n(s,t)$$

Diff eqn $$\frac{d}{dz}\Sigma_4(s,t,z) = 2s\int_0^1 dx\int_0^x dy\ (\Sigma_3(s,t',z)+\Sigma_3(t',s,z))|_{t'=xt+yu}$$

$$\Sigma_4(s,t,z)=\Sigma_3(s,t,z)+S_3(s,t)z^3 \qquad \Sigma(s,t,z)=z^{-2}\Sigma_3(s,t,z)$$

$$\boxed{\frac{d}{dz}\Sigma(s,t,z) = s-\frac{2}{z}\Sigma(s,t,z)+2s\int_0^1 dx\int_0^x dy\ (\Sigma(s,t',z)+\Sigma(t',s,z))|_{t'=xt+yu}}$$

All loop Exact Recurrence Relation

D=8 N=1

s-channel term $S_n(s,t)$ **t-channel term** $T_n(s,t)$ $T_n(s,t)=S_n(t,s)$

Exact relation for ALL diagrams

$$nS_n(s,t) = -2s^2\int_0^1 dx \int_0^x dy\ y(1-x)\ \left(S_{n-1}(s,t')+T_{n-1}(s,t')\right)|_{t'=tx+yu}$$
$$+\quad s^4\int_0^1 dx\ x^2(1-x)^2\sum_{k=1}^{n-2}\sum_{p=0}^{2k-2}\frac{1}{p!(p+2)!}\ \frac{d^p}{dt'^p}(S_k(s,t')+T_k(s,t'))\ \times$$
$$\times\frac{d^p}{dt'^p}(S_{n-1-k}(s,t')+T_{n-1-k}(s,t'))|_{t'=-sx}\ (tsx(1-x))^p$$

$$S_1=\frac{1}{12},\ T_1=\frac{1}{12}$$

summation $\Sigma_3(s,t,z)=\Sigma_1(s,t,z)-S_2(s,t)z^2+S_1(s,t)z,\ \Sigma_2(s,t,z)=\Sigma_1(s,t,z)+S_1(s,t)z$

Diff eqn

$$\frac{d}{dz}\Sigma(s,t,z) = -\frac{1}{12}+2s^2\int_0^1 dx\int_0^x dy\ y(1-x)\ (\Sigma(s,t',z)+\Sigma(t',s,z))|_{t'=tx+yu}$$
$$-s^4\int_0^1 dx\ x^2(1-x)^2\sum_{p=0}^{\infty}\frac{1}{p!(p+2)!}\left(\frac{d^p}{dt'^p}(\Sigma(s,t',z)+\Sigma(t',s,z))|_{t'=-sx}\right)^2\ (tsx(1-x))^p.$$

All loop Exact Recurrence Relation

D=10 N=1

s-channel term $S_n(s,t)$ **t-channel term** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

Exact relation for ALL diagrams

$$nS_n(s,t) = -s^3 \int_0^1 dx \int_0^x dy \; y^2(1-x)^2 \; (S_{n-1}(s,t') + T_{n-1}(s,t'))|_{t'=tx+yu}$$

$$+ \quad s^5 \int_0^1 dx \; x^3(1-x)^3 \sum_{k=1}^{n-2} \sum_{p=0}^{3k-2} \frac{1}{p!(p+3)!} \frac{d^p}{dt'^p}(S_k(s,t') + T_k(s,t')) \times$$

$$S_1 = \frac{s}{5!}, \; T_1 = \frac{t}{5!} \qquad \times \frac{d^p}{dt'^p}(S_{n-1-k}(s,t') + T_{n-1-k}(s,t'))|_{t'=-sx} \; (tsx(1-x))^p$$

summation

$$\Sigma_3(s,t,z) = \Sigma_1(s,t,z) - S_2(s,t)z^2 + S_1(s,t)z, \;\; \Sigma_2(s,t,z) = \Sigma_1(s,t,z) + S_1(s,t)z$$

Diff eqn

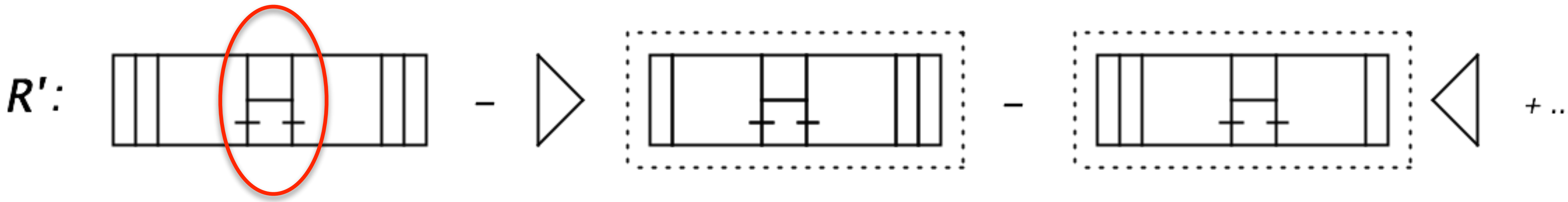
$$\frac{d}{dz}\Sigma(s,t,z) = -\frac{s}{5!} + s^3 \int_0^1 dx \int_0^x dy \; y^2(1-x)^2 \; (\Sigma(s,t',z) + \Sigma(t',s,z))|_{t'=tx+yu}$$

$$-s^5 \int_0^1 dx \; x^3(1-x)^3 \sum_{p=0}^{\infty} \frac{1}{p!(p+3)!} (\frac{d^p}{dt'^p}(\Sigma(s,t',z) + \Sigma(t',s,z))|_{t'=-sx})^2 \; (tsx(1-x))^p$$

R-operation and Recurrence Relation

D=6 N=2

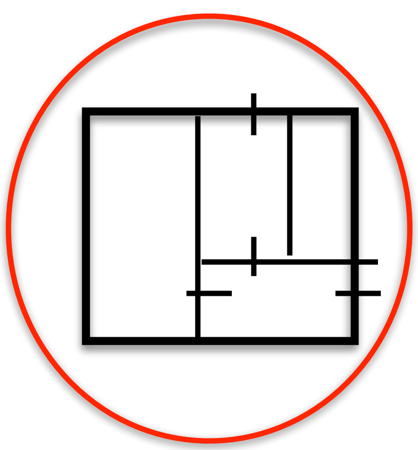
Horizontal boxes + tennis court



A_n $\qquad\qquad$ A_{n-1}

$$nA_n = -A_{n-1} \quad \longrightarrow \quad A_n = (-1)^n \frac{2}{n!} \qquad (-g^2 s)^n$$

Horizontal boxes + double tennis court



$$nA^t_n = -\frac{1}{3}A^t_{n-1}, \qquad nA^s_n = -A^s_{n-1} + \frac{1}{3}A^t_{n-1}$$

$$A^t_n = \frac{(-1)^n}{3^{n-3}}\frac{1}{n!}, \qquad A^s_n = \frac{1}{2}\frac{(-1)^n}{3^{n-3}}\frac{1}{n!} - \frac{1}{2}(-1)^n\frac{1}{n!}$$

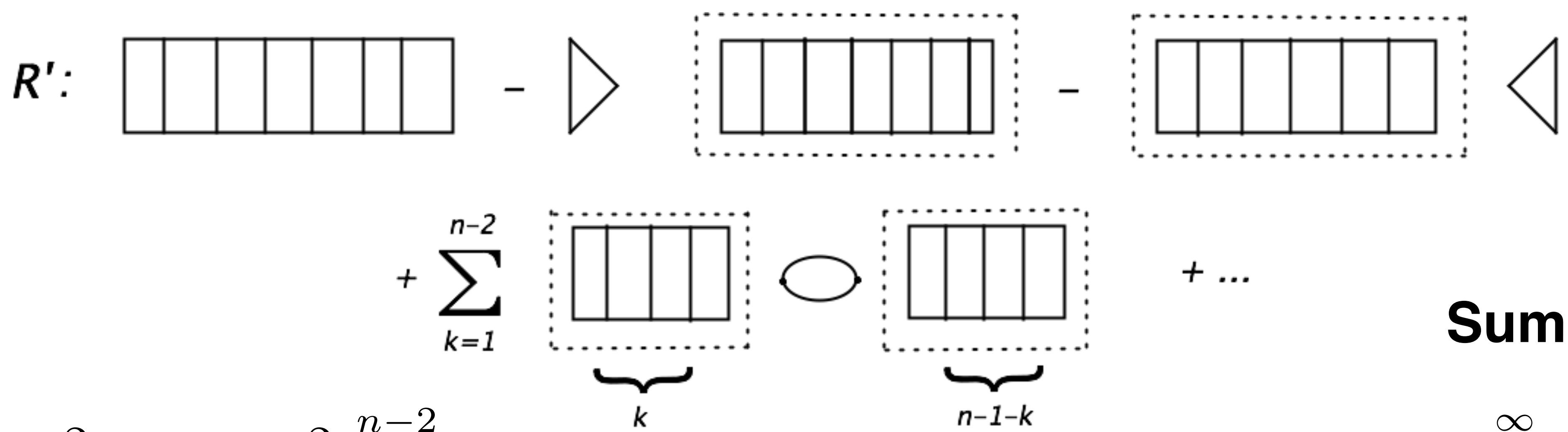
$$(-g^2 s)^{n-1}(-g^2 t) \qquad (-g^2 s)^n$$

- **Similar relations one can get for all other series**
- **All of them have 1/n! behavior**
- **Number of these series group as n!**

R-operation and Recurrence Relation

D=8 N=1

Horizontal boxes



Summation

$$nA_n = -\frac{2}{4!}A_{n-1} + \frac{2}{5!}\sum_{k=1}^{n-2} A_k A_{n-1-k}, \quad n \geq 3$$

$$\Sigma_m(z) = \sum_{n=m}^{\infty} A_n(-z)^n$$

$$-\frac{d}{dz}\Sigma_3 = -\frac{2}{4!}\Sigma_2 + \frac{2}{5!}\Sigma_1\Sigma_1. \quad \Sigma_3 = \Sigma_1 + A_1 z - A_2 z^2, \quad \Sigma_2 = \Sigma_1 + A_1 z, \quad A_1 = \frac{1}{3!}, \quad A_2 = -\frac{1}{3!4!}$$

$\Sigma \equiv \Sigma_1$

Diff eqn

$$\boxed{\Sigma' = -\frac{1}{3!} + \frac{2}{4!}\Sigma - \frac{2}{5!}\Sigma^2}$$

$$\boxed{\Sigma(z) = -\sqrt{5/3}\,\frac{4\tan[z/(8\sqrt{15})]}{1 - \tan[z/(8\sqrt{15})]\sqrt{5/3}}}$$

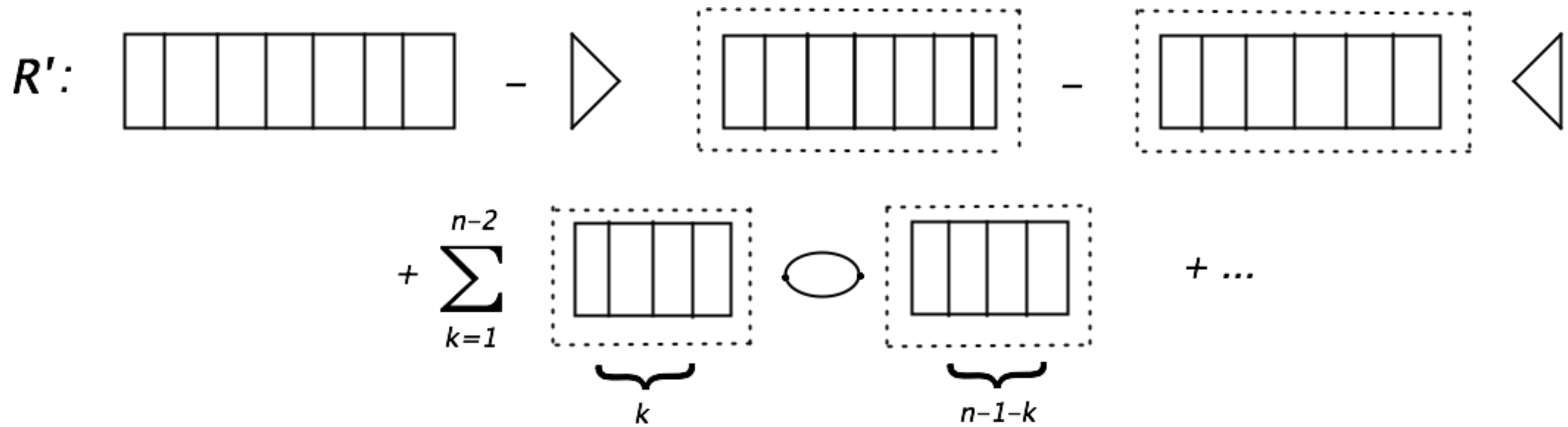
$$z = g^2 s^2/\epsilon$$

$$\Sigma(z) = -(z/6 + z^2/144 + z^3/2880 + 7z^4/414720 + \dots)$$

R-operation and Recurrence Relation

D=10 N=1

Horizontal boxes



$$
\begin{aligned}
nA_n^t &= -2\frac{2}{7!}A_{n-1}^t + \frac{1}{3\cdot 7!}\sum_{k=1}^{n-2} A_k^t A_{n-1-k}^t, \\
nA_n^s &= -2\left[\frac{1}{3\cdot 5!}A_{n-1}^s - \frac{6}{7!}A_{n-1}^t\right] \\
&+ \frac{3}{7!}\sum_{k=1}^{n-2}\left(2A_k^s A_{n-1-k}^s - A_k^s A_{n-1-k}^t - A_k^t A_{n-1-k}^s + \frac{5}{9}A_k^t A_{n-1-k}^t\right)
\end{aligned}
$$

$$A_1^s = A_1^t = 1/5!$$

The Fixed Point and Finiteness

D=6 N=2

Diff eqn for the sum of two channels

$$\frac{d}{dz}(\Sigma(s,t,z)+\Sigma(t,s,z)) = (s+t) - \frac{2}{z}[\Sigma(s,t,z)+\Sigma(t,s,z)]$$
$$+2s\int_0^1 dx \int_0^x dy\ [\Sigma(s,t',z)+\Sigma(t',s,z)]|_{t'=xt+yu}$$
$$+2t\int_0^1 dx \int_0^x dy\ [\Sigma(s',t,z)+\Sigma(t,s',z)]|_{s'=xs+yu}$$

The fixed point
$\epsilon \to 0$

$$\boxed{\Sigma(s,t,\infty)+\Sigma(t,s,\infty) = -1}$$

Finite value

Stable for s+t=-u<0

Unstable for s+t=-u>0

Finiteness:

$$\left.\begin{array}{l}\textbf{s-t channel}\quad \textbf{s+t<0}\\ \textbf{s-u channel}\quad \textbf{s+u<0}\\ \textbf{t-u channel}\quad \textbf{t+u<0}\end{array}\right\}\ \textbf{incompatible since s+t+u=0}$$

Summary

D=6 N=2   D=8 N=1   D=10 N=1

- **The UV divergences for the on-shell scattering amplitudes DO NOT CANCEL in any given order of PT**

- **The recurrence relations allow one to calculate the leading UV divergences in ALL orders of PT algebraically**

- **The sum of the leading UV divergences to ALL orders obeys the linear (D=6) or nonlinear (D=8,10) differential equation**

- **This equation possesses the fixed point. The STABLE fixed point would imply the FINITENESS of the theory when** $\epsilon \to +0$

- **Further calculations to test conjectured UV behaviour are highly desirable!**

- **String theory predictions (D=10, N=1 case) are desirable as well.**

Some Speculations

- It might mean that in nonrenormalizable theories the finite number of PT terms has no meaning while the full theory exists.
- That would imply that severe UV divergences present in any given order of PT are actually artifacts of the weak coupling expansion.

- If this is true, one may try to apply the same arguments to quantum gravity. This would mean that one should not be confused by nonrenormalizability of PT in quantum gravity.
- It may well be that the full theory is meaningful, PT is just not applicable here.

- In order to understand the nonrenormalizable theories one has to find an alternative dual description.
- The result of an alternative approach might be quite different from the PT one.

Some Speculations

D=6 N=2

- Equation for the total sum has a fixed point $\boxed{\Sigma(s,t,\infty)+\Sigma(t,s,\infty)=-1}$
- It is stable when $\epsilon \to +0$ in t-channel and unstable in s-channel
- Having in mind all channels in full amplitude the fixed point appears to be UNSTABLE

D=8 N=1 D=10 N=1

- Due to non-linearity of equation the fixed point analysis is complicated
- Existence of a fixed point does not contradict the equation
- Example of the horizontal boxes demonstrates that the limit $\epsilon \to +0$ might be similar to a gauge theory in D=4