Properties of HS equations in $d=4$

(in colaboration with N. Misuna and M. Vasiliev)

- Motivation $\longrightarrow$ HS AdS/CFT

— 2002 Klebanov – Polyakov
Sezgin – Sundell

$$\begin{tikzcd}
\text{HS} \arrow[r, leftrightarrow, "\text{dual}"] & O(N) \arrow[dl] \arrow[dr] & \\
\text{free} & & \text{critical}
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & \text{free} \\
 & & \text{boson} \arrow[ur] \arrow[dr] \arrow[dd, leftrightarrow, "+CS"] & \\
\text{HS} \arrow[r] & O(N) \arrow[ur, "a"] \arrow[dr, "b"'] & & \text{critical} \\
 & & \text{fermion} \arrow[ur] \arrow[dr] & \\
 & & & \text{free}
\end{tikzcd}$$

Hard to check: no action principle is available

- 2009, Giombi - Yin :

3pt functions match ! (from equations of motion)

- 2011, 2012, Maldacena - Zhiboedov

Slightly broken HS symmetry $\longrightarrow \langle\ \rangle_{3pt} = \frac{1}{\sqrt{N}}\left(a_B \langle\ \rangle_b + a_F \langle\ \rangle_f + a_{odd} \langle\ \rangle_{odd}\right)$

- Quantum level : determinants, free energy, anomalies

[Some remarkable cancelations take place]

No reference to details of nonlinear theory !

Open problem : <u>bulk tests</u> of the correspondence

HS equations in $d=4$

Principles that fix Vasiliev equations unknown.

Linear equations: Proper vacuum is AdS (ω, e) $\begin{cases} d\omega + \omega\wedge\omega = \Lambda\, e\wedge e \\ de + \omega\wedge e = 0 \end{cases}$

in $d=4$ $o(3,2) \sim sp(4,R) \longrightarrow$ oscillator realization

$Y_A = (y_\alpha, \bar{y}_{\dot\alpha})$ $\alpha = 1,2$ $[Y_A, Y_B]_* = 2i\,\epsilon_{AB}$ $T_{AB} = Y_A Y_B$ – generators

$$W_0 = \Omega_{AB} Y^A Y^B = \omega_{\alpha\beta}\, y^\alpha y^\beta + \bar\omega_{\dot\alpha\dot\beta}\, \bar y^{\dot\alpha} \bar y^{\dot\beta} + 2 e_{\alpha\dot\alpha}\, y^\alpha \bar y^{\dot\alpha}; \qquad dW_0 + W_0 * W_0 = 0$$

free scalar $\Box\phi = -4\Lambda\phi \longleftrightarrow C(Y|x) = \sum C_{\alpha(n),\dot\alpha(n)} (y^\alpha)^n (\bar y^{\dot\alpha})^n$

$$dC + W_0 * C - C * \pi(W_0) = 0$$

twisted-adjoint representation

$\pi(y, \bar y) = (-y, \bar y)$

$$\boxed{\begin{aligned} &dW_0 + W_0 * W_0 = 0 \\ &dC + [W_0, C]_{tw} = 0 \\ &d\omega + [W_0, \omega] = (e\wedge e)^{\alpha\beta} \frac{\partial^2}{\partial y^\alpha \partial y^\beta} C(y,0) + c.c. \end{aligned}}$$

$$\begin{cases} d\omega + \omega * \omega = F_2(\omega, C) \\ dC + \omega * C - C * \pi(\omega) = F_1(\omega, C) \end{cases}$$

$d^2 = 0 \Rightarrow$ consistency + gauge invariance

embed into flat connection?

$$\begin{cases} dW + W * W = 0 \\ dB + W * B - B * \pi(W) = 0 \end{cases}$$

$\omega(Y) \longrightarrow W(Z, Y)$ $\quad [Z, Y] = 0$

$C(Y) \longrightarrow B(Z, Y)$ $\quad [Y, Y] = -[Z, Z] \sim 1$

embedding via external connection $\quad d + W(Z, Y) + S_A \theta^A$

$\{dx, dx\} = \{dx, \theta\} = \{\theta, \theta\} = 0$

space-time 1-form $\quad$ external 1-form

$$\boxed{\begin{array}{ll} dW + W * W = 0 & \\ dS + [W, S] = 0 & S * S = -i(\theta_A \wedge \theta^A + \eta B * \gamma + \bar{\eta} B * \bar{\gamma}) \\ dB + [W, B]_{tw} = 0 & \gamma = k\varkappa\, \theta_\alpha \wedge \theta^\alpha, \quad \varkappa = e^{i z_\alpha y^\alpha} \\ [B, S]_{tw} = 0 & \bar{\gamma} = \bar{k} \bar{\varkappa}\, \bar{\theta}_{\dot\alpha} \wedge \bar{\theta}^{\dot\alpha}, \quad \bar{\varkappa} = e^{i \bar{z}_{\dot\alpha} \bar{y}^{\dot\alpha}} \end{array}}$$

Perturbation theory

$$\begin{cases} dW + W * W = 0 \\ dB + [W, B]_{tw} = 0 \\ dS_\alpha + [W, S_\alpha] = 0 \\ [S_\alpha, S_\beta] = -2i\epsilon_{\alpha\beta}(1 + \eta B * \varkappa) \\ [S_\alpha, B]_{tw} = 0 \end{cases}$$

Vacuum: $W = \Omega_{AB} Y^A Y^B$

$B = 0, \quad S_A = Z_A$

1st order: $[S^{(0)}, f(Y,Z)] \sim [Z_A, f] = -2i \dfrac{\partial}{\partial Z^A} f$

1) $[Z_A, B^{(1)}] = 0, \quad B^{(1)}(Z,Y) = C(Y) \longrightarrow dC + [W_0, C]_{tw} = 0$

2) $[S^{(0)}, S^{(1)}] = \text{''}, \quad \dfrac{\partial}{\partial Z^\alpha} S^{(1)\alpha} = \eta\, C * \varkappa \longrightarrow S^{(1)}_\alpha = \eta \displaystyle\int_0^1 dt\, t\, Z_\alpha\, C(-tZ, \bar{Y})\, e^{itZ_\beta y^\beta}$

3) $dS^{(1)} + [W_0, S^{(1)}] + [W^{(1)}, S^{(0)}] = 0$

$$dS^{(1)}_\alpha - 2i \frac{\partial}{\partial Z^\alpha} W^{(1)} + [W_0, S^{(1)}_\alpha] = 0; \quad W^{(1)} = \frac{i}{2}\int_0^1 dt\, Z^\alpha \left(dS^{(1)}_\alpha + [W_0, S^{(1)}_\alpha]\right) + \omega(Y)$$

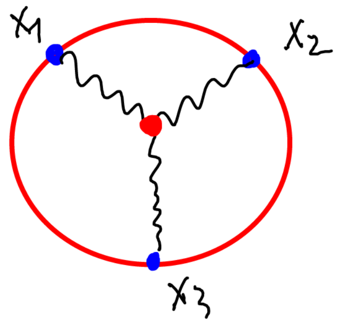
4) $dW^{(1)} + [W_0, W^{(1)}] = 0 \rightarrow d\omega + [W_0, \omega] = -\dfrac{i}{4}(e \wedge e)^{\alpha\beta} \dfrac{\partial^2}{\partial y^\alpha \partial y^\beta} C(y, 0) + c.c.$

Vasiliev eqs $\longrightarrow$ HS equations $\begin{cases} d\omega + \omega * \omega = \mathcal{V}_2(\omega, \omega, C, \dots C, \dots) \\ dC + [\omega, C]_{*} = \mathcal{V}_1(\omega, C, \dots C, \dots) \end{cases}$

Generic scheme: $\left\{ \begin{array}{l} z\text{-dependence for } B \\ \downarrow \\ \text{solving for } S \\ \downarrow \\ z\text{-dependence of } W \\ \downarrow \\ \text{equations on } \omega \text{ and } C \end{array} \right.$

Available tests of HS AdS/CFT from the bulk

① Giombi-Yin: 3pt functions were extracted from $C(Y|x)$ to the 2nd order



+ "correct" correlation functions
+ C is related to conserved currents
- C is not gauge invariant
- no divergencies control

2) Colombo - Sundell

correlation functions from supertrace $\longleftrightarrow$ $\mathrm{str}(B)$

+ allows one to reproduce <u>all</u> N-point correlators in free theory via

$$\langle \ldots \rangle = \mathrm{str}(B_1 * \ldots * B_N)$$

− str does not exist at nonlinear level

3) Skvortsov and friends

Reconstruction of known cubic vertices from Vasiliev eqs.

?? puzzling - divergent coefficients

4) Vasiliev : Invariant functionals

Vasiliev construction

$$\mathcal{W} = d + W + S \rightarrow$$

$$\boxed{\begin{array}{l} \mathcal{W} * \mathcal{W} = -i\left(\theta_A \wedge \theta^A + \eta B * \gamma + \bar{\eta} B * \bar{\gamma}\right) \\ [\mathcal{W}, B]_* = 0 \end{array}}$$

$$\gamma = k\varkappa\, \theta_\alpha \wedge \theta^\alpha$$

$$\bar{\gamma} = \bar{k}\bar{\varkappa}\, \bar{\theta}_{\dot\alpha} \wedge \bar{\theta}^{\dot\alpha}$$

Introducing higher forms

$$\mathcal{W} = d + \sum_{i=0}^{1} W^{i,1-i} + \sum_{i=0}^{3} W^{i,3-i}$$

$$W^{i,j} = W_{...}(dx)^i(\theta)^j$$

$$W^{1,0} = W\ ; \quad W^{0,1} = S$$

$$B \rightarrow B + \sum_{i=0}^{2} B^{i,2-i}$$

$$\boxed{\begin{array}{l} \mathcal{W} * \mathcal{W} = -i\left(\theta_A \wedge \theta^A + \eta B * \gamma + \bar{\eta} B * \bar{\gamma} + g\, \gamma * \bar{\gamma}\, \delta^4(\theta)\right) \\ [\mathcal{W}, B] = 0 \end{array}}$$

Admits gauge invariant 4-form

$$\mathcal{L} \sim (W * W)\Big|_{Y,Z,\theta=0}$$

$$\sim \mathrm{str}(W * W)\Big|_{\theta=0}$$

Back to perturbation theory

Vacuum: $B_{(0)}=0$; $W^{1,0}_{(0)}=W_{AdS}$; $W^{0,1}_{(0)}=z_A\theta^A$, $W^{i,3-i}_{(0)}$ – ?

$$W^{3}_{(0)}=\int_0^1 dt\, t^3 e^{itz_AY^A} z^A\frac{\partial}{\partial\theta^A}\,\delta^{(4)}\left(\theta_A+i(1-t)\,\omega_A{}^B z_B\right)$$ ω_{AB} – AdS connection

1st order: $\underbrace{W^{0,1}\to B\to W^{1,0}}_{\text{1 and 0 - forms}}$; $\underbrace{W^{0,3}\to B^{0,2}\to W^{1,2}\to B^{1,1}\to W^{2,1}\to B^{2,0}\to W^{3,0}}_{\text{3 and 2 - forms}}$

$\to$ very hard [hurt me plenty]

2nd order: insane [Nightmare]

Call for new methods for perturbation.

At all stages of perturbative analysis one faces with

$$df + [\omega_{AdS}, f] - d_Z f = J \qquad (\Delta f = J)$$

equiv. $\quad D f - d_Z f = J \; ; \quad D^2 = 0 , \quad \{D, d_Z\} = 0$

$$f \to f^{i,j} \; ; \quad f_n = f^{n, N-n}$$

$$\boxed{\begin{aligned} -d_Z f_0 &= J_0 \\ -d_Z f_1 &= J_1 - D f_0 \\ &\cdots\cdots \\ -d_Z f_N &= J_N - D f_{N-1} \\ D\hat{f} &= J_{N+1} - D f_N \end{aligned}}$$

Example: Let $\quad D = 0 \Rightarrow d_Z f = J$

$$f = d_Z^{-1} J + d_Z \epsilon + \hat{f} \; ; \quad d_Z d_Z^{-1} \neq 1 \; ; \quad d_Z d_Z^{-1} d_Z = d_Z$$

$$d_Z^{-1} J(Z, Y; \theta) = Z^A \frac{\partial}{\partial \theta^A} \int_0^1 \frac{dt}{t} J(tZ, Y; t\theta)$$

$$\boxed{\{d_Z, d_Z^{-1}\} = 1 - \hat{h}} \; , \quad \hat{h} - \text{projector to } Z\text{-cohomol.}$$

$$\hat{h} f(Z, Y; \theta) = f(0, Y; 0)$$

Analogously, $f = \Delta^{-1} J + \Delta \mathcal{E} + \hat{f}$; $D\hat{f} = J_{N+1} - D f_N$

$$\Delta^{-1} J = \sum_{n=0}^{N} f_n = - \sum_{n=0}^{N \to \infty} (d_z^{-1} D)^n d_z^{-1} J = - d_z^{-1} \frac{1}{1 - D d_z^{-1}} J$$

$$D\hat{f} = \mathcal{H} J \, ; \qquad \mathcal{H} = \hat{h} \frac{1}{1 - D d_z^{-1}} \, ; \qquad \boxed{\{\Delta, \Delta^{-1}\} = 1 - \mathcal{H}}$$

resolution of identity

Adjoint case

$$D \sim \omega^{AB} Y_A \frac{\partial}{\partial Y^B} + \omega^{AB} \frac{\partial^2}{\partial Y^A \partial Z^B}$$

Using that $\displaystyle\iint_{[0,1]^2} dt\, dt'\, t^m t'^n A(tt'x) = \frac{1}{m-n} \int_0^1 dt\, (t^m - t^n) A(tx)$, $m \neq n$

$$\boxed{\Delta^{-1} J = - Z^A \frac{\partial}{\partial \theta^A} \int_0^1 dt\, J\left(tZ, Y_A + \frac{1-t}{t} \omega_A{}^B \frac{\partial}{\partial \theta^B} ; t\theta\right)}$$

$$\boxed{\mathcal{H} J(Z, Y; \theta) = J\left(0, Y_A + \omega_A{}^B \frac{\partial}{\partial \theta^B} ; \theta\right)\Big|_{\theta=0}}$$

Examples

- Standard HS equations to the 1st order

$$\begin{cases} dW + W * W = -i\,(\theta_A \wedge \theta^A + \eta\, B * \gamma + \bar{\eta}\, B * \bar{\gamma}) \\ dB + [W, B] = 0 \,, \qquad \gamma = \varkappa\, \theta_\alpha \wedge \theta^\alpha = e^{i z_\alpha y^\alpha}\, \theta_\beta \wedge \theta^\beta \end{cases}$$

$$\Delta_{ad} W^{(1)} = -i\,(\eta\, C * \gamma + \bar{\eta}\, C * \bar{\gamma}) \;;$$

$$W^{(1)} = \text{z-dependent} + \text{cohomology } (\omega(Y))$$

$$\text{z-dependent} \downarrow \quad \Delta^{-1}_{ad}(\ldots) \qquad\qquad \text{cohomology} \downarrow \quad D\omega(Y|x) = \mathcal{H}(\eta\, C * \gamma + c.c.)$$

$$C * \gamma = C * e^{i z_\alpha y^\alpha}\, \theta_\beta \wedge \theta^\beta = C(-z, \bar{y})\, \delta^2(\theta)$$

$$\mathcal{H}(C * \gamma) = C\Big(0, \bar{y}_{\dot\alpha} + e^{\beta}{}_{\dot\alpha} \frac{\partial}{\partial \theta^\beta}\Big)\, \theta^\gamma \wedge \theta_\gamma \Big|_{\theta=0} \longrightarrow (e \wedge e)^{\dot\alpha\dot\beta}\, \frac{\partial^2}{\partial \bar{y}^{\dot\alpha}\, \partial \bar{y}^{\dot\beta}}\, C(0, \bar{y})$$

Vacuum of extended system

$$\begin{cases} dW + W * W = -i\left(\theta_A \wedge \theta^A + \eta B * \gamma + \bar{\eta}\, B * \bar{\gamma} + g\, \gamma * \bar{\gamma}\ \delta^4(\theta)\right) \\ dB + [W, B] = 0 \end{cases}$$

$B_{(0)} = 0\,, \qquad W^{1,0}_{(0)} = W_{AdS}\,, \qquad W^{0,1}_{(0)} = z^A \theta_A \quad ; \qquad W^{i,\,3-i}_{(0)}$ – ?

$$\Delta_{ad} W = -ig\, e^{i z_A Y^A}\, \delta^4(\theta) \quad ;$$

$$W = g\, \Delta^{-1}_{ad}\left(e^{i z_A Y^A}\, \delta^4(\theta)\right) = -z^A \frac{\partial}{\partial \theta^A} \int_0^1 dt\, t^3\, e^{it z_A\left(Y^A + \frac{1-t}{t}\omega^{AB}\frac{\partial}{\partial \theta^B}\right)}\, \delta^4(\theta)$$

$$= -z^A \frac{\partial}{\partial \theta^A} \int_0^1 dt\, t^3\, e^{it z_A Y^A}\, \delta^4\left(\theta_A + i(1-t)\,\omega_A{}^B z_B\right)$$

Conclusion

- A new method to perturbation theory for Vasiliev HS equations is proposed. It allows one reproducing vertices in

$$\begin{cases} d\omega + \omega * \omega = \mathcal{V}(\omega,\omega,C) + \mathcal{V}(\omega,\omega,C,C) + \dots \\ dC + [\omega, C]_{*tw} = \mathcal{V}(\omega,C,C) + \mathcal{V}(\omega,C,C,C) + \dots \end{cases}$$

in extremely economic way. More importantly, it gives an extra control for field redefinition.

- The method makes the problem of evaluating invariant functionals tractable. We hope to apply it for testing the HS AdS/CFT.