Higher Spin fields and charges in periodic twistor space

Y. Goncharov, M. Vasiliev

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Result

For HS fields desribed by $Sp(2M)$-invariant unfolded equations, for Y-periodic solutions, the complete set of non-trivial conserved charges is constructed. Leftover global symmetry is presented.

Outline

1) Sp(2M)-invariant formulation.
2) Rank-one equations – fields, rank-two equations – currents, symmetry transformations. Charges.
3) Periodic case: fields, currents, symmetries. Theta functions.
4) Charges: integration surfaces.
5) Charges: current cohomology.

$Sp(2M)$-invariant formulation

Fronsdal '85:

- $Sp(8)$ acting on HS multiplet in $4d$ Minkowski
- Generalized spacetime $\mathcal{M}_4$ with coordinates X^{AB}, $A, B = \overline{1, 4}$ (4×4 real symmetric matrices), $\dim \mathcal{M}_4 = 10$.

$Sp(2M)$-invariant formulation

General $Sp(2M)$-invariant approach:

- Generalized spacetime $\mathcal{M}_M$ $(M \geq 2)$, coordinates X^{AB}, $A, B = \overline{1, M}$, $\dim \mathcal{M}_4 = \frac{M(M+1)}{2}$.
- fields $C\,(X)\,, C_A\,(X) \in \mathbb{R}$ + eom: bosons and fermions
- $C = C^+ + C^-$, $C^\pm \in \mathbb{C}$ – positive- and negative-frequency parts

Vasiliev '01,'03

Gelfond, Vasiliev '08

Florakis, Sorokin, Tsulaia '14

Rank-one fields

- Rank-one equation:

$$\left(\frac{\partial}{\partial X^{AB}} \pm i \frac{\partial^2}{\partial Y^A \partial Y^B}\right) C^{\pm}(Y|X) = 0$$

- General solution (basis $\theta_\xi(Y|X) = e^{i\left(\xi_A X^{AB} \xi_B + \xi_A Y^A\right)}$):

$$C(Y|X) = \int d^M \xi \, c(\xi) \, \theta_\xi(Y|X) \, ,$$

- $\mathcal{D}$-function:

$$\mathcal{D}(Y|X) = \frac{1}{(2\pi)^M} \int d^M \xi \, \theta_\xi(Y|X) \, ,$$

$$\mathcal{D}^{\pm}(Y|0) = \delta(Y) \, .$$

Symmetries

All symmetries are represented in terms of twistor variables:

- Action on Y-variables by Y^A and $Z_A = \frac{\partial}{\partial Y^A}$ – Heisenberg algebra H_M: $\left[Z_A, Y^B\right] = \delta_A^B$.
- Symmetry transformation $\eta(Y, Z|X)\ C^{\pm}(Y|X)$:

$$\left(\frac{\partial}{\partial X^{AB}} + i\left[\frac{\partial^2}{\partial Y^A Y^B}, \cdot\right]\right)\eta = 0$$

- Covariant oscillators

$$\mathcal{A}^C = Y^C - 2iX^{CB}Z_B,\ \mathcal{B}_C = Z_C$$

Covariant oscillators

- Covariant oscillators – elements of H_M:

$$\left[\mathcal{B}_A, \mathcal{A}^B\right] = \delta_A^B, \ \left[\mathcal{B}_A, \mathcal{B}^C\right] = 0, \ \left[\mathcal{A}^B, \mathcal{A}^C\right] = 0$$

- Any symmetry transformation $\eta\,(Y, Z|X) = \eta(\mathcal{A}, \mathcal{B})$
- Action on basis vectors $\theta_\xi\,(Y|X) = \exp\left(i\,\xi X \xi + i\,\xi Y\right)$:

$$\mathcal{B}_C\,\theta_\xi = i\xi\,\theta_\xi, \quad \mathcal{A}^C\,\theta_\xi = -i\frac{\partial}{\partial \xi_C}\,\theta_\xi$$

- Fock-like representation of basis vectors:

$$\mathcal{B}_C\,\theta_0 = 0, \ e^{i\xi\,\mathcal{A}}\,\theta_0 = \theta_\xi$$

Rank-two fields. Bilinear fields

Doubling of variables $Y \to Y_1, Y_2$:

- Rank-two unfolded equation:

$$\left(\frac{\partial}{\partial X^{AB}} + i \frac{\partial^2}{\partial {Y_1}^A \partial {Y_1}^B} - i \frac{\partial^2}{\partial {Y_2}^A \partial {Y_2}^B} \right) J\left(Y_1, Y_2 | X\right) = 0$$

- Symmetry transformations $\eta\left(Y_{1,2}, Z^{1,2} | X\right) J\left(Y_{1,2} | X\right)$:

$$\eta = \eta\left(\mathcal{A}_{1,2}, \mathcal{B}^{1,2}\right)$$

- Bilinear field:

$$J\left(Y_1, Y_2 | X\right) = \eta\left(\mathcal{A}_{1,2}, \mathcal{B}^{1,2}\right) C^{+}\left(Y_1 | X\right) C^{-}\left(Y_2 | X\right)$$

Bilinear currents. Charges

Rank-two fields allows to construct M-form closed on-shell:

- Bilinear current $d\Omega_\eta = 0$:

$$\Omega_\eta = \left(dV^A + i\, dX^{AB} \frac{\partial}{\partial U^B} \right)^M \eta\, C^+ (V - U|X)\, C^- (V + U|X)$$

- Conserved charge for a Cauchi surface Σ

$$Q_\eta = \int_\Sigma \Omega_\eta$$

- Non-zero charges for $\eta\,(\mathcal{P}^1, \mathcal{Q}_2)$ (Vasiliev'13) $\left[\mathcal{P}^i, \mathcal{Q}_j\right] = \delta^i_j$

$$\mathcal{P}^1 = \mathcal{B}^2 - \mathcal{B}^1, \qquad \mathcal{P}^2 = \mathcal{B}^2 + \mathcal{B}^1,$$
$$\mathcal{Q}_1 = \frac{1}{2}\,(\mathcal{A}_2 - \mathcal{A}_1)\,, \quad \mathcal{Q}_2 = \frac{1}{2}\,(\mathcal{A}_2 + \mathcal{A}_1)\,.$$

Periodic solutions

- Series over $\theta_n(Y|X) = \exp(i\,nXn + i\,nY)$, $n \in \mathbb{Z}^M$

$$C(Y|X) = \sum_{n\in\mathbb{Z}^M} c_n\,\theta_n(Y|X)\,,\ \ C(Y+2\pi|X) = C(Y|X)$$

- Induced X-periodicity: $\pi(1+\delta_{AB})$ for X^{AB}
- $\mathcal{D}$-function is Riemann theta-function ()

$$\mathcal{D}(Y|X) = \frac{1}{(2\pi)^M}\sum_n e^{i(nXn+nY)} = \theta(Y|X)$$

- Action of covariant oscillators gives theta-function with characteristics

$$e^{b\,\mathcal{B}}e^{ia\,\mathcal{A}}\,\theta(Y|X) = \theta\left[{a \atop b}\right](Y|X) \equiv e^{i((n+a)X(n+a)+(n+a)(Y+b))}$$

Symmetry generators

- $\mathcal{A}$ does not act properly on $\{\theta_n\} \Rightarrow$ generators of symmetry transformations are $e^{i\mathcal{A}^C}$, $\mathcal{B}_C$. The most general form of a symmetry transformation:

$$\eta = \eta\left(e^{i\mathcal{A}}, \mathcal{B}\right)$$

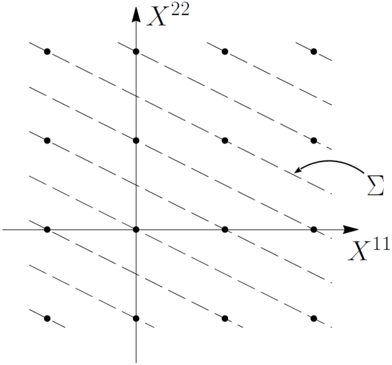
- General symmetry parameter for bilinear currents:

$$\eta = \eta\left(\mathcal{P}^{1,2}, e^{i\mathcal{Q}_{1,2}}\right)$$

Integration cycles

Periodic solutions are functions on a torus $\mathcal{M}_M \times \mathbb{R}^M_{(Y)}/L$, where L – lattice of periods

- Integration surfaces $\Sigma \subset \mathcal{M}_M \times \mathbb{R}^M$ are invariant wrt the shifts by the elements of L



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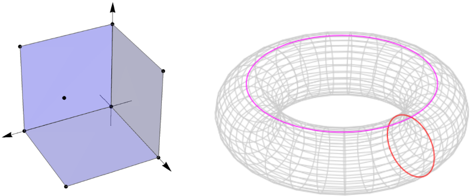
- Integration over any surface Σ reduces to integration over *cells* – M-dimensional surfaces being products of M fundamental cycles



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- Integration over any surface Σ reduces to integration over *cells* – M-dimensional surfaces being products of M fundamental cycles (figure)
- A cell lying in twistor space – a *fundamental cell*.

Current cohomology

Reducing dependence on irrelevant oscillators in

$$\eta = \eta\left(\mathcal{P}^{1,2}, e^{i\mathcal{Q}_{1,2}}\right)$$

- By adding an exact form it is straightforward to get

$$\eta = \eta\left(\mathcal{P}^{1}, e^{i\mathcal{Q}_{1,2}}\right)$$

- For integration over the basis cell $\mathcal{Q}_1$-dependence can be reduced:

$$\eta = \eta\left(\mathcal{P}^{1}, e^{i\mathcal{Q}_{2}}\right)$$

- For any surface Σ and any η (16) the corresponding charge can be obtaided by integration over the basis cell with

$$\tilde{\eta}\left(\mathcal{P}^{1}, \mathcal{Q}_{1,2}\right) = p_{\Sigma}\left(i\mathcal{P}^{1}\right) \star \eta\left(\mathcal{P}^{1}, \mathcal{Q}_{1,2}\right)$$

Conclusion

- Non-zero charges are parametrized by generators

$$\mathcal{P}^1, \; e^{i\mathcal{Q}_2}$$

  what is analogous to the non-periodic case
- On the contrary to the non-periodic case, integration over the basis cell is implied:

All charges obtained from higher cycles correspond to higher symmetries on the basis cycle