Covariant structure constants for deformed oscillator algebra $Aq\,(2, \nu)$

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Outline

Deformed Oscillators

$$[y_\alpha, y_\beta] = 2i\,(1+\nu\mathcal{K}) \quad \alpha,\beta = +,-, \tag{1}$$

$$\{\mathcal{K}, y_\alpha\} = 0, \quad \mathcal{K}^2 = 1, \tag{2}$$

where $\nu \in \mathbb{R}$.

- Harmonic Oscillator. Wigner 50

$$H = \frac{i}{8}\,(y_+ y_- + y_- y_+)\,, \quad [H, y_+] = \frac{1}{2} y_+, \quad [H, y_-] = -\frac{1}{2} y_-; \tag{3}$$

- Higher spin algebra. Vasiliev 89,91
- Non-linear HS equations. Vasiliev 90,03
- AdS_3/CFT_2 correspondence. Gaberdiel and Gopakumar 13

sl_2 algebra and $hs\,[\lambda]$

Bilinears obey sl_2 commutation relations

$$J_0 = -\frac{i}{8}\left(y_+ y_- + y_- y_+\right), \quad J_{\pm 1} = \frac{i}{4} y_\pm^2 \tag{4}$$

$$\left[J_m, J_n\right] = (m-n)\, J_{m+n}, \quad m, n = -1, 0 + 1 \tag{5}$$

Casimir operator

$$C_2 = C_2\,(\nu) = \frac{1}{16}\left(\nu^2 - 2\nu\mathcal{K} - 3\right) \tag{6}$$

Higher spin algebra

$$hs\,[\lambda] = \frac{U\,(sl_2)}{C_2 - \frac{1}{4}\left(\lambda^2 - 1\right)\mathbf{1}}, \quad shs\,[\lambda] = \frac{U\,(osp\,(1|2))}{C - \frac{1}{4}\lambda\,(\lambda - 1)\,\mathbf{1}} \tag{7}$$

Structure constants. Fradkin and Linetsky 91

$Aq\left(2,\nu\right)$

$shs\left[\lambda\right]$ – is the Lie algebra associated with associative algebra $Aq\left(2,\nu\right)$. Generic element of $Aq\left(2,\nu\right)$

$$f\left(y,\mathcal{K}\right)=\sum_{n=0}^{\infty}f_{A}^{\alpha(n)}\mathcal{K}^{A}\underbrace{y_{\alpha}\ldots y_{\alpha}}_{n}, \tag{8}$$

where $f_{A}^{\alpha(n)}=f_{A}^{\alpha_{1}\ldots\alpha_{n}}$ – totally symmetric tensor of rank n with respect to α indices, $A=0,1$.

$Aq\,(2,\nu)$

Product of two generic elements of $Aq\,(2,\nu)$ is

$$f\,(y,\mathcal{K}) * g\,(y,\mathcal{K}) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f_A^{\alpha(n)} g_B^{\beta(m)} \sum_{p=0}^{min(m,n)} \tilde{A}^{AB}\,(m,n,p,\nu\mathcal{K})\,(\epsilon_{\alpha\beta})^p \underbrace{\overbrace{y_\alpha\ldots y_\alpha}^{m-p}\overbrace{y_\beta\ldots y_\beta}^{n-p}}_{m+n-2p}, \tag{9}$$

where $\tilde{A}\,(m,n,p,\nu\mathcal{K})$ – structure constants for the product of two monomials with powers m and n. Oscillators on the r.h.s. are totally symmetrized.

- Vasiliev 89
- Pope, Romans, Shen 90
- Joung, Mkrtchyan 14

Joung, Mkrtchyan product

$$L(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} \xi^{\alpha_1 \tilde{\alpha}_1} \ldots \xi^{\alpha_n \tilde{\alpha}_n} y_{(\alpha_1} y_{\tilde{\alpha}_1} \ldots y_{\alpha_n} y_{\tilde{\alpha}_n)} \tag{10}$$

where $\xi^{\alpha_1 \tilde{\alpha}_1}$ is symmetric sl_2 tensor. The product is

$$L(\xi) * L(\eta) = \sum_{n=0}^{\infty} {}_2F_1 \left(\begin{matrix} n + \frac{3-\nu}{2} & n + \frac{1+\nu}{2} \\ & n + \frac{3}{2} \end{matrix} ; \; -\frac{1}{4}\phi \right) \frac{1}{n!} \times$$
$$\times \, \zeta^{\alpha_1 \tilde{\alpha}_1} \ldots \zeta^{\alpha_n \tilde{\alpha}_n} y_{(\alpha_1} y_{\tilde{\alpha}_1} \ldots y_{\alpha_n} y_{\tilde{\alpha}_n)}, \tag{11}$$

where $\zeta^{\alpha\beta} = \xi^{\alpha\beta} + \eta^{\alpha\beta} + \xi^{\alpha\gamma}\eta^{\delta\beta}\epsilon_{\gamma\delta}$ and $\phi = \xi^{\alpha_1\alpha_2}\eta^{\beta_1\beta_2}\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}$.

Multiplication with projector

$$\Pi_{\pm} = \frac{1 \pm \mathcal{K}}{2} \tag{12}$$

$$f(y) * g(y) := f(y) * g(y)\,\Pi_{+} \tag{13}$$

$$f^{\alpha(m)} \underbrace{y_\alpha \dots y_\alpha}_{m} * g^{\beta(n)} \underbrace{y_\beta \dots y_\beta}_{n} =$$

$$= f^{\alpha(m)} g^{\beta(n)} \sum_{p=0}^{min(m,n)} A(m,n,p,\nu)\,(\epsilon_{\alpha\beta})^p \underbrace{\overbrace{y_\alpha \dots y_\alpha}^{m-p}\overbrace{y_\beta \dots y_\beta}^{n-p}}_{m+n-2p} \tag{14}$$

Even × Even. Conformal Basis

$$V_n^s = (-1)^{s-1-n} \frac{(n+s-1)!}{(2s-2)!} \left[J_-, \ldots \left[J_-, \left[J_-, J_+^{s-1}\right]\right]\right]. \tag{15}$$

$$V_m^s * V_n^t = \frac{1}{2} \sum_{u=1}^{s+t-1} g_u^{st}(m,n,\lambda) V_{m+n}^{s+t-u}, \tag{16}$$

with

$$g_u^{st}(m,n,\lambda) = \frac{\left(\frac{1}{4}\right)^{u-2}}{2(u-1)!} {}_4F_3 \left[\begin{matrix} \frac{1}{2}+\lambda & \frac{1}{2}-\lambda & \frac{2-u}{2} & \frac{1-u}{2} \\ \frac{3-2s}{2} & \frac{3-2t}{2} & \frac{1}{2}+s+t-u \end{matrix} ; 1\right] \times$$
$$\sum_{k=0}^{u-1} (-1)^k \binom{u-1}{k} (s-1-m)_{u-1-k} (s-1-m)_k (t-1+n)_k (t-1-n)_{u-1-k}, \tag{17}$$

$${}_4F_3 \left[\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{matrix} ; x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n (a_4)_n}{(b_1)_n (b_2)_n (b_3)_n} \frac{x^n}{n!}, \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{18}$$

Even $\times$ Even. Covariant Basis

$$A(m,n,p,\nu) = i^p \frac{m!n!}{(m-p)!\,(n-p)!p!}\,{}_4F_3\left[\begin{matrix} 1-\frac{\nu}{2} & \frac{\nu}{2} & \frac{-p}{2} & \frac{1-p}{2} \\ \frac{1-m}{2} & \frac{1-n}{2} & \frac{m+n-2p+3}{2} & \end{matrix};1\right], \tag{19}$$

Saalschutzian transform.

If $a_1 + a_2 + a_3 + a_4 + 1 = b_1 + b_2 + b_3$ and $(-a_4) \in \mathbb{N}$, then

$$\begin{aligned} {}_4F_3\left[\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & \end{matrix};1\right] &= \frac{(b_2-a_1)_{-a_4}\,(b_3-a_1)_{-a_4}}{(b_2)_{-a_4}\,(b_3)_{-a_4}} \times \\ &\times {}_4F_3\left[\begin{matrix} a_1 & b_1-a_2 & b_1-a_3 & a_4 \\ b_1 & a_1-b_2+a_4+1 & a_1-b_3+a_4+1 & \end{matrix};1\right] \end{aligned} \tag{20}$$

Associativity Condition

$$f^{\alpha(m)}\underbrace{y_\alpha\dots y_\alpha}_{m}*g^{\beta(n)}\underbrace{y_\beta\dots y_\beta}_{n}=f^{\alpha(m)}\underbrace{y_\alpha\dots y_\alpha}_{m}*g^{\beta(n)}\underbrace{y_\beta\dots y_\beta}_{n-2}*y_\beta y_\beta \qquad (21)$$

$$f^{\alpha(m)}g^{\beta(n)}\sum_{p=0}^{min(m,n)}A\left(m,n,p,\nu\right)\left(\epsilon_{\alpha\beta}\right)^p\underbrace{\overbrace{y_{(\alpha}\dots y_\alpha}^{m-p}\overbrace{y_\beta\dots y_{\beta)}}^{n-p}}_{m+n-2p}=$$

$$=f^{\alpha(m)}g^{\beta(n-2)\gamma_1\gamma_2}\left(\sum_{p=0}^{min(m,n-2)}A\left(m,n-2,p,\nu\right)\left(\epsilon_{\alpha\beta}\right)^p\underbrace{\overbrace{y_{(\alpha}\dots y_\alpha}^{m-p}\overbrace{y_\beta\dots y_{\beta)}}^{n-2-p}}_{m+n-2-2p}\right)y_{\gamma_1}y_{\gamma_2} \qquad (22)$$

$$A\left(m,n,p,\nu\right)=A\left(m,n-2,p,\nu\right)+2i\left(m-p+1\right)A\left(m,n-2,p-1,\nu\right)+$$
$$+i^2A\left(m,n-2,p-2,\nu\right)\left(m-p+2\right)\left(m-p+1\right)\frac{m+n-2p+3-\nu}{m+n-2p+3}\,\frac{m+n-2p+1+\nu}{m+n-2p+1}, \qquad (23)$$

Other structure constants

Odd × Odd

$$B(m+1,n+1,p,\nu) = A(m,n,p,-\nu) + i(m+n-2p+3+\nu)A(m,n,p-1,-\nu) + \\ + i^2(m-p+2)(n-p+2)\frac{m+n-2p+5+\nu}{m+n-2p+5}\frac{m+n-2p+3+\nu}{m+n-2p+3} \times A(m,n,p-2,\nu), \tag{24}$$

Even × Odd

$$C(m,n+1,p,\nu) = A(m,n,p,-\nu) + \\ + i(m-p+1)\frac{m+n-2p+3+\nu}{m+n-2p+3}A(m,n,p-1,-\nu), \tag{25}$$

Odd × Even

$$D(m+1,n,p,\nu) = A(m,n,p,\nu) + \\ + i(n-p+1)\frac{m+n-2p+3-\nu}{m+n-2p+3}A(m,n,p-1,\nu). \tag{26}$$

Associativity Verification

$$F\left(m,n,p,\nu\right) = {}_4F_3\left[\begin{matrix} 1-\frac{\nu}{2} & \frac{\nu}{2} & \frac{-p}{2} & \frac{1-p}{2} \\ \frac{1-m}{2} & \frac{1-n}{2} & \frac{m+n-2p+3}{2} & \end{matrix} ;1\right] \tag{27}$$

Associativity condition

$$\begin{aligned} \frac{n!}{(n-p)!p!}F\left(m,n,p,\nu\right) = \frac{(n-2)!}{(n-p-2)!p!}F\left(m,n-2,p,\nu\right) + \\ + 2\frac{(n-2)!}{(n-p-1)!\,(p-1)!}F\left(m,n-2,p-1,\nu\right) + \\ + \frac{(n-2)!}{(n-p)!\,(p-2)!}\frac{m+n-2p+3-\nu}{m+n-2p+3}\frac{m+n-2p+1+\nu}{m+n-2p+1}F\left(m,n-2,p-2,\nu\right). \end{aligned} \tag{28}$$

$$F\left(m,n-2,\ldots,\nu\right) = \text{Coeff.}\, F\left(m,n,p,\nu\right) + \text{extra terms} \tag{29}$$

1st Hypergeometry $F\left(m,n-2,p,\nu\right)$

$$F\left(m,n-2,p,\nu\right)=\sum_{q=0}^{\infty}\frac{\left(1-\frac{\nu}{2}\right)_q\left(\frac{\nu}{2}\right)_q\left(\frac{1-p}{2}\right)_q\left(-\frac{p}{2}\right)_q}{\left(\frac{1-m}{2}\right)_q\left(\frac{1-n}{2}+1\right)_q\left(\frac{m+n-2p+3}{2}-1\right)_q q!}. \tag{30}$$

$$\left(a\right)_q=\frac{\Gamma\left(a+q\right)}{\Gamma\left(a\right)} \tag{31}$$

$$\begin{aligned}F\left(m,n-2,p,\nu\right)=&\frac{m+2n-2p}{m+n-2p+1}\,{}_4F_3\left[\begin{matrix}1-\frac{\nu}{2} & \frac{\nu}{2} & \frac{-p}{2} & \frac{1-p}{2}\\ \frac{1-m}{2} & \frac{1-n}{2}+1 & \frac{m+n-2p+3}{2} & \end{matrix};1\right]-\\ &-\frac{n-1}{m+n-2p+1}\underbrace{{}_4F_3\left[\begin{matrix}1-\frac{\nu}{2} & \frac{\nu}{2} & \frac{-p}{2} & \frac{1-p}{2}\\ \frac{1-m}{2} & \frac{1-n}{2} & \frac{m+n-2p+3}{2} & \end{matrix};1\right]}_{F(m,n,p,\nu)}.\end{aligned} \tag{32}$$

3rd Hypergeometry $F\left(m, n-2, p-2, \nu\right)$

Suppose p is even $p = 2N$, then Saalschutzian transform

$$F\left(m,n-2,p-2,\nu\right) = {}_4F_3\left[\begin{matrix} 1-\frac{\nu}{2} & \frac{\nu}{2} & 1-\frac{p}{2} & 1+\frac{1-p}{2} \\ \frac{1-m}{2} & \frac{1-n}{2}+1 & \frac{m+n-2p+3}{2}+1 & \end{matrix};1\right] =$$

$$= \frac{\left(\frac{p-m}{2}-1\right)_{N-1}}{\left(\frac{1-m}{2}\right)_{N-1}} \frac{\left(\frac{p-n}{2}\right)_{N-1}}{\left(\frac{1-n}{2}+1\right)_{N-1}} \times$$
$$\times\, {}_4F_3\left[\begin{matrix} \frac{m+n-2p+3+\nu}{2} & \frac{m+n-2p+5-\nu}{2} & 1-\frac{p}{2} & 1+\frac{1-p}{2} \\ \frac{m}{2}-p+3 & \frac{n}{2}-p+2 & \frac{m+n-2p+5}{2} & \end{matrix};1\right]. \tag{33}$$

3rd Hypergeometry $F\left(m,n-2,p-2,\nu\right)$

$$F\left(m,n-2,p-2,\nu\right)=\frac{\left(\frac{p-m}{2}-1\right)_{N-1}}{\left(\frac{1-m}{2}\right)_{N-1}}\frac{\left(\frac{p-n}{2}\right)_{N-1}}{\left(\frac{1-n}{2}+1\right)_{N-1}}\frac{\left(\frac{m+n-2p+3}{2}\right)\left(\frac{m}{2}-p+2\right)}{\left(\frac{m+n-2p+3-\nu}{2}\right)\left(\frac{m+n-2p+1+\nu}{2}\right)}\times$$
$$\left(\frac{m}{2}-p+1\right)\frac{\left(\frac{n}{2}-p+1\right)}{\left(\frac{1-p}{2}\right)\left(\frac{-p}{2}\right)}\times$$
$$\left({}_4F_3\left[\begin{matrix}\frac{m+n-2p+1+\nu}{2} & \frac{m+n-2p+3-\nu}{2} & -\frac{p}{2} & \frac{1-p}{2}\\ \frac{m}{2}-p+1 & \frac{n}{2}-p+1 & \frac{m+n-2p+3}{2} & ;1\end{matrix}\right]-\right.$$
$$\left.-{}_4F_3\left[\begin{matrix}\frac{m+n-2p+1+\nu}{2} & \frac{m+n-2p+3-\nu}{2} & -\frac{p}{2} & \frac{1-p}{2}\\ \frac{m}{2}-p+2 & \frac{n}{2}-p+1 & \frac{m+n-2p+3}{2} & ;1\end{matrix}\right]\right). \quad (34)$$

2nd Hypergeometry $F\left(m,n-2,p-1,\nu\right)$

$$F\left(m,n-2,p-1,\nu\right)=A\;F\left(m,n-2,p-1,\nu\right)+B\;F\left(m,n-2,p-1,\nu\right) \tag{35}$$

$$A+B=1 \tag{36}$$

- A-term. Shifts like in $F\left(m,n-2,p,\nu\right)$
- B-term. Saalschutzian transform like in $F\left(m,n-2,p-2,\nu\right)$

To cancel extra terms

$$A=\frac{m+2n-2p}{m+n-2p+1},\quad B=1-\frac{m+2n-2p}{m+n-2p+1}. \tag{37}$$

Conclusion

- All structure constants for $Aq(2,\nu)$ in covariant basis are found.
- The fact that structure constants contain Saalschutzian hypergeometric functions is crucial.
- Starting point to search for deformed oscillator star product.