Holographic interpretation of toroidal blocks in the semicalssical limit

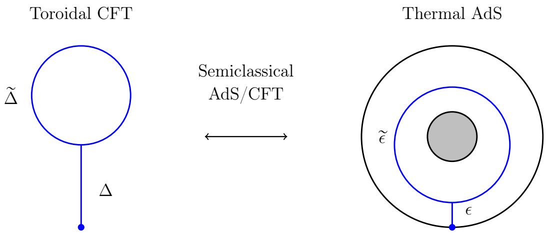
Konstantin Alkalaev

I.E.Tamm Theory Department, P.N.Lebedev Physical Institute

K.Alkalaev, V.Belavin, arXiv:1603.08440

Moscow 2016

Summary of results



$$-f^{lin} = S_{thermal} + \tilde{\epsilon}\, S_{loop} + \epsilon\, S_{leg} \ ,$$

Plan

- Toroidal 1-point blocks
- Tadpoles on thermal AdS
- AdS/CFT and modular invariance

Quantum toroidal block

- map from complex plane to cylinder
- identify edge states

The (holomorphic, quantum) 1-point conformal block of the primary field $\phi_\Delta(z)$

$$\mathcal{V}(\Delta, \tilde{\Delta}, c|q) = q^{\tilde{\Delta}-c/24} \sum_{n=0}^{\infty} q^n \mathcal{V}_n(\Delta, \tilde{\Delta}, c) \ ,$$

where

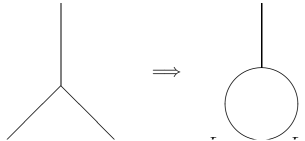
$$q = e^{2\pi i \hat{\tau}} \ ,$$

is the elliptic parameter on a torus with the modulus $\hat{\tau}$, and the expansion coefficients are

$$\mathcal{V}_n(\Delta, \tilde{\Delta}, c) = \frac{1}{\langle \tilde{\Delta} | \phi_\Delta | \tilde{\Delta} \rangle} \sum_{n=|M|=|N|} \frac{\langle \tilde{\Delta}, M | \phi_\Delta | N, \tilde{\Delta} \rangle}{\langle \tilde{\Delta}, M | N, \tilde{\Delta} \rangle} \ .$$

Here, $|\tilde{\Delta}, M\rangle$ are the M-th level descendant vectors in the Verma module generated from the primary state $|\tilde{\Delta}\rangle$. As usual, $|M|$ denotes the minus sum of Virasoro generator indices. Note that the 1-point conformal block is independent of the insertion point z.

Tadpole graph



Classical toroidal block

In the $c \to \infty$ limit the 1-point *classical* block on a torus (see, e.g. *Piatek 2013*)

$$\mathcal{V}(\Delta, \tilde{\Delta}|q) = \exp\Big[-\frac{c}{6} f(\epsilon, \tilde{\epsilon}|q)\Big] \;, \qquad \Delta = c\epsilon/6 \;, \quad \tilde{\Delta} = c\tilde{\epsilon}/6 \;,$$

where

$$f(\epsilon, \tilde{\epsilon}|q) = (\tilde{\epsilon} - 1/4)\log q + \sum_{n=1}^{\infty} q^n \mathrm{f}_n(\epsilon, \tilde{\epsilon}) \;, \qquad \text{where} \qquad \mathrm{f}_1 = \frac{\epsilon^2}{2\tilde{\epsilon}} \;, \qquad \ldots$$

Perturbative regime. Introduce the *lightness* parameter

$$\delta = \epsilon/\tilde{\epsilon} < 1 \;.$$

Then, changing from $(\epsilon, \tilde{\epsilon})$ to $(\delta, \tilde{\epsilon})$ we represent the classical conformal block as a double series expansion in q and δ keeping terms at most linear in $\tilde{\epsilon}$,

$$f(\epsilon, \tilde{\epsilon}|q) = f^{lin}(\delta, \tilde{\epsilon}|q) + \mathcal{O}(\tilde{\epsilon}^2) \;, \qquad \text{where} \qquad f^{lin}(\delta, \tilde{\epsilon}|q) = (\tilde{\epsilon} - 1/4)\log q + \tilde{\epsilon}\sum_{n=1}^{\infty} f_n(q)\delta^{2n} \;,$$

with expansion coefficients written in a closed form as

$$f_n = \varkappa_n q^n (1-q)^{-2n+1} \left(q^{n-1} + \gamma_{n-2} q^{n-2} + \ldots + \gamma_1\right) \;,$$

where $\gamma_i = (-)^i \binom{2n+i-1}{i}$, $i = 0, 1, \ldots, n-1$ are binomial coefficients, and $\varkappa_i$ are some constants.

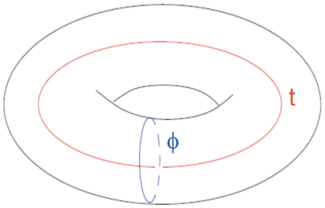
- The 1-point block can be considered as a small deformation of the 0-point block where the deformation parameter is identified with the external dimension.
- Even orders δ^{2n} !

Thermal AdS

Euclidean thermal AdS space has the metric

$$ds^2 = -\tau^2 \left(1 + \frac{r^2}{l^2}\right) dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\phi^2 \ ,$$

where τ is the pure imaginary modular parameter, and coordinates $t \sim t + 2\pi$, $\phi \sim \phi + 2\pi$, $r \geq 0$.

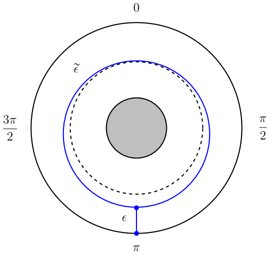


Topologically, the thermal AdS is a solid torus with time running along the non-contractible cycle.

- $(\mathrm{Re}\,\tau) = 0$ — special (rectangular) torus!

Dual interpretation

Constant angle slice: *annulus*



$$S_{total} = S_{thermal} + \tilde{\epsilon}\, S_{loop} + \epsilon\, S_{leg} \ ,$$

- $\Delta = 0$: 1-point conformal block $\Longrightarrow$ 0-point conformal block (Virasoro character). The corresponding graph is a constant radius circle going along the origin $r = 0$.

Total action is

$$S_{total} = S_{thermal} + \tilde{\epsilon}\, S_{loop} \ ,$$

where $S_{thermal} = i\pi\tau/2$ in terms of the rescaled central charge $k = c/6$ (*Maldacena & Strominger 1998*), while the circumference is $S_{loop} = -2\pi i\tau$ we find that

$$S_{total}\Big|_{\epsilon=0} = -2\pi i\ (\tilde{\epsilon} - 1/4)\ \tau\ \ :=\ -(\tilde{\epsilon} - 1/4)\log q = -f^{lin}(\epsilon, \tilde{\epsilon}|q)\Big|_{\epsilon=0} \ .$$

Worldline approach

The worldline action of a single massive particle with $m \sim \epsilon$ is

$$S = \epsilon \int_{\lambda'}^{\lambda''} d\lambda \sqrt{g_{tt}\dot{t}^2 + g_{\phi\phi}\dot{\phi}^2 + g_{\rho\rho}\dot{\rho}^2} \,, \qquad ds^2 = -\tau^2 \left(1 + \frac{r^2}{l^2}\right) dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\phi^2$$

- Boundary coordinates t and ϕ are cyclic — a constant angle annulus (ρ, t).
- The normalization condition

$$|g_{mn}(x)\dot{x}^m \dot{x}^n| = 1 \;: \qquad \dot{r} = \pm\sqrt{r^2 - s^2 + 1} \,, \qquad s \equiv i\,\frac{|p_t|}{\tau} \,.$$

- The circumference of the loop can be calculated by using the definition of the time momentum $p_t = g_{tt}\dot{t}$. Representing the loop as two semi-loops we find that

$$\dot{t} = \frac{i}{\tau}\,\frac{s}{1 + r^2} \;: \qquad S_{loop} = \frac{2\tau}{is} \int_0^{\pi} dt \big(1 + r^2(t)\big) \,,$$

  where the radial deviation $r(t)$ is defined from the normalization condition.
- The boundary condition

$$r(0) = 0 \,, \qquad r(\pi) = \rho \,,$$

  where ρ is the vertex radial position. If the loop is a constant radius circle then we find that $s^2 = 1 + r^2$, and, therefore, $S_{loop} = -2\pi i\tau s$. For $r = 0$ the length is $S_{loop} = -2\pi i\tau$.
- The time momentum of the external leg is $s = 0$: radial direction

$$S_{leg} = \int_{\rho}^{\Lambda} \frac{dr}{\sqrt{1 + r^2}} = -\mathrm{ArcSinh}\,\rho \; + \; \text{infinite cutoff } (\Lambda)$$

  The cutoff parameter Λ is introduced to regularize the conformal boundary position.

Equilibrium equation

Minimizing the vertex configuration of three lines

$$\tilde{\epsilon}\,(p^1_m + p^2_m) + \epsilon p^0_m = 0\ ,$$

where

- $p^{1,2}$ are the ingoing/outgoing intermediate momenta and p^0 is an external momentum
- Spacetime index takes just two values $m = (t, r)$
- Any closed curve has $|p^1_m| = |p^2_m|$ while overall sings can be different. Indeed, their relative sign $\pm$ depends on whether we take $m = t$ or $m = r$.

Time component is given by

$$\tilde{\epsilon}(s_1 - s_2) + \epsilon s_0 = 0\ , \qquad s \equiv i\,\frac{|p_t|}{\tau}$$

As the loop has $s_1 = s_2 \equiv s$ we find out that $s_0 = 0$ (it goes along the radial direction!). Radial component is more interesting

$$\tilde{\epsilon}(\dot{r}_1 + \dot{r}_2) - \epsilon \dot{r}_0 = 0\ ,$$

where $\dot{r}$ can be found from the normalization condition. Since $r_1 = r_2$ we find that $\delta \dot{r}_0 = 2\dot{r}_1$, where $\delta = \epsilon/\tilde{\epsilon}$. Then, the vertex position ρ is expressed in terms of the loop momentum s as

$$\rho = \sqrt{\frac{s^2}{1 - \delta^2/4} - 1}\ .$$

We see that if the external field is decoupled $\delta = 0$ then there is the following solution

$$\delta = 0\ : \qquad s = 1\ , \qquad \rho = 0\ .$$

Radial deviation equation

To find how the loop radial deviation evolves in time we use the time loop momentum $p_t = g_{tt}\dot{t}$ and recall that $\dot{r}$ is given by the normalization condition. Their ratio

$$\frac{dt}{dr} = \frac{i}{\tau}\frac{s}{(1+r^2)\sqrt{r^2-s^2+1}} \, , \qquad s \equiv i\,\frac{|p_t|}{\tau}$$

It integrates to *radial deviation equation*

$$e^{-2i\tau t}(r^2+1)(s^2-1) = -r(2s\sqrt{r^2-s^2+1}+rs^2+r)+s^2-1 \, ,$$

where the time dependence is given only via the exponential. Note that the modular parameter enters only through this exponential and this is how the elliptic parameter appears in the final expressions.

Solving the radial deviation equation as $r = r(t)$ can be problematic — no exact solutions!

Instead, we develop the deformation method which allows us to find s and $r(t)$ as a power series in δ with the leading term given by the seed solution $s = 1$ and $r(t) = 0$.

- *Our strategy:* Solve the equilibrium and radial deviation equations perturbatively in δ!

Perturbative expansion

We consider the tadpole graph perturbatively starting from the seed solution corresponding to the loop of constant radius and adding small interaction with the external leg. In this way we are led to calculate the geodesic length function

$$L = \tilde{\epsilon}\, S_{loop}(\tau, \epsilon, \tilde{\epsilon}) + \epsilon\, S_{leg}(\tau, \epsilon, \tilde{\epsilon}) \ ,$$

Both lengths can be represented as power series in the lightness parameter δ as

$$S_{loop} = \sum_{n=0}^{\infty} S_{loop}^{(n)}(\tau)\delta^n \ , \qquad S_{leg} = \sum_{n=0}^{\infty} S_{leg}^{(n)}(\tau)\delta^n \ ,$$

where $S_{loop}^{(0)}(\tau) = -2\pi i \tau$ and $S_{leg}^{(0)}(\tau) = 0$. Noting that $\tilde{\epsilon}\delta^n = \epsilon\delta^{n-1}$ the length function is given by

$$L = -2\pi i \tilde{\epsilon}\tau + \tilde{\epsilon} \sum_{n=1}^{\infty} \left[S_{loop}^{(n)}(\tau) + S_{leg}^{(n-1)}(\tau) \right] \delta^n \ .$$

Comparing with the linearized block expression we find the condition

$$S_{loop}^{(n)}(\tau) + S_{leg}^{(n-1)}(\tau) = 0 \ , \qquad \text{for} \qquad n = 2k+1 \ , \quad k = 0, 1, 2, \ldots \ .$$

Time momentum and the radial deviation along with the vertex position are expanded as

$$s = \sum_{n=0}^{\infty} s_{2n}\, \delta^{2n} \ , \qquad r(t) = \sum_{n=0}^{\infty} r_{2n+1}(t)\, \delta^{2n+1} \ , \qquad \rho = r(\pi)$$

where the seed values are $s_0 = 1$ and $r_0(t) = 0$.

More calculations ...

We find

$$r_1(t) = \frac{1}{2}\operatorname{sech}(-i\pi\tau)\sinh(-i\tau t)\ , \qquad r_3(t) = \frac{1}{16}\operatorname{sech}^3(-i\pi\tau)\sinh(-i\tau t)\cosh^2(-i\tau t)\ , \ldots$$

and the loop momentum corrections

$$s_2 = -\frac{1}{8}\operatorname{sech}^2(-i\pi\tau)\ , \qquad s_4 = -\frac{1}{128}\operatorname{sech}^4(-i\pi\tau)\ , \ldots$$

Substituting $t = \pi$ we find the vertex position corrections

$$\rho = g(\delta)\tanh(-i\pi\tau)\ , \qquad \text{where} \qquad g(\delta) = \frac{1}{2}\delta + \frac{1}{16}\delta^2 + \frac{3}{256}\delta^3 + \ldots\ .$$

Now, it is straightforward to find lower order corrections to the loop length and the leg length:

$$S^{(2)}_{loop}(\tau) = -\frac{1}{4}\tanh(-i\pi\tau)\ , \qquad S^{(1)}_{leg}(\tau) = \frac{1}{2}\tanh(-i\pi\tau)\ ,$$

Identifying the modular parameters as $\tau \to \tau + \frac{1}{2}$ in the conformal parametrization

$$S^{(2)}_{loop}(q) := f_1(q) + \frac{1}{4}\ , \qquad S^{(1)}_{leg}(q) := -2f_1(q) - \frac{1}{2}\ ,$$

and so modulo an additive constant the length function in the first nontrivial order is

$$L(q) := -\tilde{\epsilon}\log q - \tilde{\epsilon}\,\delta^2 f_1(q) + \mathcal{O}(\delta^4)\ .$$

Adding the thermal AdS action term we find out that $S_{thermal}(q) + L(\epsilon,\tilde{\epsilon}|q) = -f(\epsilon,\tilde{\epsilon}|q)$.

...and final interpretation

We observe that the AdS/CFT correspondence holds for

- $\tau_{CFT} = \tau_{AdS} + \frac{1}{2}$ (exact correspondence)
- $\mathrm{Im}\,\tau_{AdS} \to \infty$ (low-temperature approximation)

Conclusions & outlooks

Conclusions

- We considered the classical 1-point conformal block on a torus and defined its linearized version: intermediate is heavier than external!
- Thermal AdS is the background and the block function is realized as the geodesic length of the tadpole + the thermal action.

Outlooks

- Monodromy approach
- Higher point blocks and their dual interpretation, including W_N symmetry.
- Holographic entropies, worldline interpretation of toroidal conformal blocks as Wilson lines of CS formulation, etc.