Perturbative analysis in higher-spin theories

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Outline

- Nonlinear HS equations
- Homotopy trick
- Adjoint case
- Twisted case
- Examples

Nonlinear HS equations

- Nonlinear HS equations for $W\left(x|Z^A, Y^A|k,\bar{k}|dx,\theta^A\right)$ and $B\left(x|Z^A, Y^A|k,\bar{k}\right)$:

$$\mathrm{d}_X W + W * W = -i\theta_\alpha \wedge \theta^\alpha \left(1+\eta B * \varkappa k\right) - i\bar{\theta}_{\dot\alpha} \wedge \bar{\theta}^{\dot\alpha}\left(1+\bar{\eta}B * \bar{\varkappa}\bar{k}\right),$$

$$\mathrm{d}_X B + W * B - B * W = 0.$$

- HS star product:

$$f * g\,(Z;Y) = \int d^4U d^4V e^{iU_A V^A} f\left(Y+U;Z+U\right) g\left(Y+V;Z-V\right).$$

- Interior Klein operator:

$$\varkappa := \exp\left(iz_\alpha y^\alpha\right), \quad \varkappa * \varkappa = 1, \quad \varkappa * f\left(z^\alpha, y^\alpha\right) = f\left(-z^\alpha, -y^\alpha\right) * \varkappa.$$

- Exterior Klein operator:

$$kk = 1, \qquad kf\left(z^\alpha, y^\alpha, \theta^\alpha\right) = f\left(-z^\alpha, -y^\alpha, -\theta^\alpha\right)k.$$

Perturbative analysis

- AdS_4 background is provided by

$$B_0 = 0, \qquad W_0 = \phi_{AdS} + Z_A \theta^A,$$

where ϕ_{AdS} - space-time 1-form of AdS_4-connection

$$\mathrm{d}_X \phi_{AdS} + \phi_{AdS} * \phi_{AdS} = 0.$$

$$\phi_{AdS} = -\frac{i}{4}\phi^{AB} Y_A Y_B = -\frac{i}{4}\left(\omega^{AB} + h^{AB}\right) Y_A Y_B,$$

- While expanding Vasiliev equations over this vacuum two types of perturbative equations arise, in the adjoint and twisted adjoint sectors,

$$\Delta_{ad} f := \mathrm{d}_X f + [\phi_{AdS}, f]_* - 2i\mathrm{d}_Z f = J,$$

$$\Delta_{tw} f := \mathrm{d}_X f - \frac{i}{4}\left[\omega^{AB} Y_A Y_B, f\right]_* - \frac{i}{4}\left\{h^{AB} Y_A Y_B, f\right\}_* - 2i\mathrm{d}_Z f = J,$$

where $\mathrm{d}_Z := \theta^A \frac{\partial}{\partial Z^A}$ and $\Delta_{ad}(fk) = (\Delta_{tw} f)\, k$.

Homotopy trick

Consider some nilpotent operator d, $\mathrm{d}^2 = 0$. If there is a homotopy operator ∂, $\partial^2 = 0$, such that

$$A := \{\mathrm{d}\,, \partial\}$$

is diagonalizable, then $H(\mathrm{d}) \subset KerA$; we can introduce projector $\hat{h}$ to $KerA$ and

$$A^* A = AA^* = Id - \hat{h}\,.$$

$$\mathrm{d}^* := \partial A^*$$

$$\{\mathrm{d}\,, \mathrm{d}^*\} + \hat{h} = Id\,.$$

This provides a general solution (if any) to $\mathrm{d}f = J$,

$$f = \mathrm{d}^* J + \mathrm{d}\epsilon + g,$$

where $g \in H(\mathrm{d})$.

Homotopy trick: de Rham

For $\mathrm{d}_Z = \theta^A \frac{\partial}{\partial Z^A}$ in trivial topology one can set

$$\partial_Z = Z^A \frac{\partial}{\partial \theta^A} \,.$$

This gives

$$A = \theta^A \frac{\partial}{\partial \theta^A} + Z^A \frac{\partial}{\partial Z^A},$$

$$A^* f(Z;Y;\theta) = \int_0^1 dt \frac{1}{t} f(tZ;Y;t\theta)\,.$$

$$\hat{h} J(Z;Y;\theta) = J(0;Y;0)\,,$$

$$\mathrm{d}_Z^* J(Z;Y;\theta) = Z^A \frac{\partial}{\partial \theta^A} \int_0^1 dt \frac{1}{t} J(tZ;Y;t\theta)\,.$$

Homotopy trick: spectral sequence

- While analysing HS equations, one faces spectral sequence

$$\Delta f := \mathrm{d}_Z f + \mathcal{D} f = J\,,$$

$$\mathrm{d}_Z^2 = 0, \qquad \{\mathrm{d}_Z, \mathcal{D}\} = 0\,, \qquad \mathcal{D}^2 = 0.$$

- Being expanded, it adds up to zigzag system in θ-degree ($f_m \sim (\theta)^m$)

$$\begin{aligned}
&\mathrm{d}_Z f_m = J_{m+1},\\
&\mathrm{d}_Z f_{m-1} = J_m - \mathcal{D} f_m,\\
&\dots\\
&\mathrm{d}_Z f_0 = J_1 - \mathcal{D} f_1,\\
&\mathcal{D} g = J_0 - \mathcal{D} f_0.
\end{aligned}$$

- Direct application of d_Z^* leads to awkward expressions with to m-fold multiple integrals.
- Can we build a proper resolution of identity to resolve Z-dependence for such system at once?

$$\{\Delta, \Delta^*\} + \mathscr{H} = Id.$$

Adjoint case

- Adjoint *AdS*-derivative:

$$\mathcal{D}_{ad} = \mathrm{d}_X + [\phi_{AdS}, \bullet]_* = \mathrm{d}_X + \phi^{AB}\left(Y_A - i\frac{\partial}{\partial Z^A}\right)\frac{\partial}{\partial Y^B}.$$

- Rewrite general adjoint equation

$$\Delta_{ad} f := -2i\theta^A \frac{\partial}{\partial Z^A} f + \mathrm{d}_X f + \phi^{AB}\left(Y_A - i\frac{\partial}{\partial Z^A}\right)\frac{\partial}{\partial Y^B} f = J$$

as

$$-2i\left(\theta^A + \frac{1}{2}\phi^{AB}\frac{\partial}{\partial Y^B}\right)\frac{\partial}{\partial Z^A} f = J - \mathcal{D}^Y_{ad} f,$$

$$\mathcal{D}^Y_{ad} := \mathrm{d}_X + \phi^{AB} Y_A \frac{\partial}{\partial Y^B}.$$

- Introduce

$$\hat{T}_{ad} = \exp\left\{\frac{1}{2}\phi^{BC}\frac{\partial^2}{\partial Y^B \partial \theta^C}\right\}$$

then

$$\left(\theta^A + \frac{1}{2}\phi^{AB}\frac{\partial}{\partial Y^B}\right)\frac{\partial}{\partial Z^A} = \hat{T}_{ad}\left(\theta^A \frac{\partial}{\partial Z^A}\right)\hat{T}^{-1}_{ad}.$$

Adjoint case

- By virtue of $\left[\hat{T}_{ad}, \mathcal{D}^Y_{ad}\right] = 0$ from

$$-2i\,\hat{T}_{ad}\left(\theta^A \frac{\partial}{\partial Z^A}\right)\hat{T}^{-1}_{ad} f = J - \mathcal{D}^Y_{ad} f$$

one gets

$$\mathrm{d}_Z \widetilde{f} = \widetilde{J} - \mathcal{D}^Y_{ad} \widetilde{f}.$$

$$\widetilde{f} := \hat{T}^{-1}_{ad} f.$$

- Its general solution is

$$\widetilde{f} = \mathrm{d}^*_Z \widetilde{J} + \widetilde{g} + \mathrm{d}_Z \widetilde{\epsilon} + \mathcal{D}^Y_{ad} \widetilde{\epsilon},$$

with arbitrary $\widetilde{\epsilon}$, and $\widetilde{g}$ solving

$$\mathcal{D}^Y_{ad} \widetilde{g} = \hat{h} \widetilde{J}.$$

Adjoint case

- In initial terms,

$$f = \hat{T}_{ad} \mathrm{d}_Z^* \hat{T}_{ad}^{-1} J + g + \Delta\epsilon,$$

$$\mathcal{D}_{ad} g = \hat{T}_{ad} \hat{h} \hat{T}_{ad}^{-1} J.$$

- Define

$$\Delta_{ad}^* J := \hat{T}_{ad} \mathrm{d}_Z^* \hat{T}_{ad}^{-1} J = -\frac{1}{2i} Z^A \frac{\partial}{\partial \theta^A} \int_0^1 dt \frac{1}{t} \exp\left(-\frac{1-t}{2t} \phi^{BC} \frac{\partial^2}{\partial Y^B \partial \theta^C} \right) J\,(tZ; Y; t\theta)\,,$$

$$\mathscr{H}_{ad} J\,(Z, Y, \theta) := \hat{T}_{ad} \hat{h} \hat{T}_{ad}^{-1} = \hat{h} \exp\left(-\frac{1}{2} \phi^{AB} \frac{\partial^2}{\partial Y^A \partial \theta^B} \right) J\,(Z; Y; \theta)\,.$$

- Then from

$$\left\{ d_Z, d_Z^* \right\} \widetilde{f} + \hat{h}\widetilde{f} = \widetilde{f}.$$

it follows that

$$\left\{ \Delta_{ad}, \Delta_{ad}^* \right\} + \mathscr{H}_{ad} = Id.$$

Twisted case

- Twisted-adjoint AdS-derivative

$$\mathcal{D}_{tw} = \mathrm{d}_X - \frac{i}{4}\left[\omega^{AB} Y_A Y_B, \bullet\right]_* - \frac{i}{4}\left\{h^{AB} Y_A Y_B, \bullet\right\}_*$$

has the form

$$\mathcal{D}_{tw} = \mathrm{d}_X + \omega^{AB}\left(Y_A - i\frac{\partial}{\partial Z^A}\right)\frac{\partial}{\partial Y^B} - \frac{i}{2}h^{AB}\left(Y_A Y_B - 2iY_A\frac{\partial}{\partial Z^B} - \frac{\partial^2}{\partial Y^A \partial Y^B} - \frac{\partial^2}{\partial Z^A \partial Z^B}\right).$$

- Now equation is

$$-2i\left(\theta^A + \frac{1}{2}\omega^{AB}\frac{\partial}{\partial Y^B} - \frac{i}{2}h^{AB}Y_B - \frac{1}{4}h^{AB}\frac{\partial}{\partial Z^B}\right)\frac{\partial}{\partial Z^A}f = J - \mathcal{D}^Y_{tw}f,$$

so we define

$$\hat{T}_{tw} = \exp\left\{\frac{1}{2}\omega^{BC}\frac{\partial}{\partial Y^B \partial \theta^C} - \frac{i}{2}h^{BC}Y_B\frac{\partial}{\partial \theta^C} - \frac{1}{4}h^{BC}\frac{\partial^2}{\partial Z^B \partial \theta^C}\right\}.$$

Twisted case

- This yields

$$f = \Delta^*_{tw} J + g + \Delta_{tw}\epsilon,$$

$$\mathcal{D}_{tw} g = \mathscr{H}_{tw} J,$$

where

$$\begin{aligned}
\Delta^*_{tw} J \quad &:= \quad -\frac{1}{2i} Z^C \frac{\partial}{\partial \theta^C} \int_0^1 dt \frac{1}{t} \exp\left\{ -\frac{i}{8}\left(\frac{1-t}{t}\right)^2 \omega^{AB} h_A{}^C \frac{\partial^2}{\partial\theta^B \partial\theta^C} + i\frac{1-t}{2t} h^{AB} Y_A \frac{\partial}{\partial\theta^B} \right\} \cdot \\
&\cdot \exp\left\{ -\frac{1-t}{2t}\omega^{AB}\frac{\partial^2}{\partial Y^A \partial\theta^B} + \frac{1-t^2}{4t^2} h^{AB} \frac{\partial^2}{\partial Z^A \partial\theta^B} \right\} J\left(tZ; Y; t\theta\right). \\
\mathscr{H}_{tw} J \quad &:= \quad \hat{h} \exp\left\{ -\frac{i}{8}\omega^{AB} h_A{}^C \frac{\partial^2}{\partial\theta^B\partial\theta^C} + \frac{i}{2} h^{AB} Y_A \frac{\partial}{\partial\theta^B} \right\} \cdot \\
&\cdot \exp\left\{ -\frac{1}{2}\omega^{AB}\frac{\partial^2}{\partial Y^A\partial\theta^B} + \frac{1}{4} h^{AB}\frac{\partial^2}{\partial Z^A\partial\theta^B} \right\} J\left(Z; Y; \theta\right).
\end{aligned}$$

- Once again from $\left\{d_Z, d_Z^*\right\}\widetilde{f} + \hat{h}\widetilde{f} = \widetilde{f}$ one finds that

$$\{\Delta_{tw}, \Delta^*_{tw}\} + \mathscr{H}_{tw} = Id.$$

Examples: free HS equations

- Vasiliev equations in the linear order

$$\Delta_{ad} W_1 = -i\eta B_1 * \gamma - i\bar{\eta} B_1 * \bar{\gamma},$$

$$\Delta_{ad} B_1 = 0,$$

where $\gamma := \theta_\alpha \theta^\alpha \varkappa k$. Physical equations are stored in Z-cohomology sector.

- According to $\{\Delta, \Delta^*\} + \mathscr{H} = Id$, $B_1 = C(y, \bar{y})\, k + c.c.$ is Z-independent and obey

$$\mathcal{D}_{tw} C(y, \bar{y}) = 0.$$

- In the sector of 1-forms we get

$$\mathcal{D}_{ad}\omega(Y) = -i\hat{h} \exp\left(-\frac{1}{2}\phi^{AB} \frac{\partial^2}{\partial Y^A \partial \theta^B}\right)(\eta B_1 * \gamma + \bar{\eta} B_1 * \bar{\gamma})$$

Elementary computation gives

$$\mathcal{D}_{ad}\omega(Y) = -\frac{i}{4}\eta h_\mu{}^{\dot{\alpha}} h^{\mu\dot{\beta}} \frac{\partial^2}{\partial \bar{y}^{\dot{\alpha}} \partial \bar{y}^{\dot{\beta}}} \left(C(0, \bar{y}) + \bar{C}(0, \bar{y})\, k\bar{k}\right) + c.c.$$

Examples: higher orders

- Acting in a similar fashion, in the second order one finds

$$\mathcal{D}_{tw} C = -\mathscr{H}_{tw} \left[W_1, B_1\right]_* ,$$

$$\mathcal{D}_{ad}\omega = -\mathscr{H}_{ad} \left(W_1 * W_1 + i\eta B_2 * \gamma + i\bar{\eta} B_2 * \bar{\gamma}\right) .$$

- Now it is not difficult to write down n-th order equations

$$\mathcal{D}_{ad}\omega\,(Y) = -\sum_{p=1}^{n-1} \mathscr{H}_{ad} \left(W_p * W_{n-p} + i\eta B_n * \gamma + i\bar{\eta} B_n * \bar{\gamma}\right) ,$$

$$\mathcal{D}_{tw} \left(Ck + \bar{C}\bar{k}\right)(Y) = -\sum_{p=1}^{n-1} \mathscr{H}_{tw} \left(W_p * B_{n-p} - B_{n-p} * W_p\right) ,$$

$$B_n\,(Z;Y) = -\sum_{p=1}^{n-1} \Delta^*_{tw} \left(W_p * B_{n-p} - B_{n-p} * W_p\right) ,$$

$$W_n\,(Z;Y) = -\sum_{p=1}^{n-1} \Delta^*_{ad} \left(W_p * W_{n-p} + i\eta B_n * \gamma + i\bar{\eta} B_n * \bar{\gamma}\right) .$$

Conclusion

A new method to perform perturbative computations in the nonlinear HS theory is proposed, based on the application of the homotopy trick to the spectral sequences that arise while expanding over AdS_4-background. All presented formulas can be relatively easily obtained from certain similarity transformations.