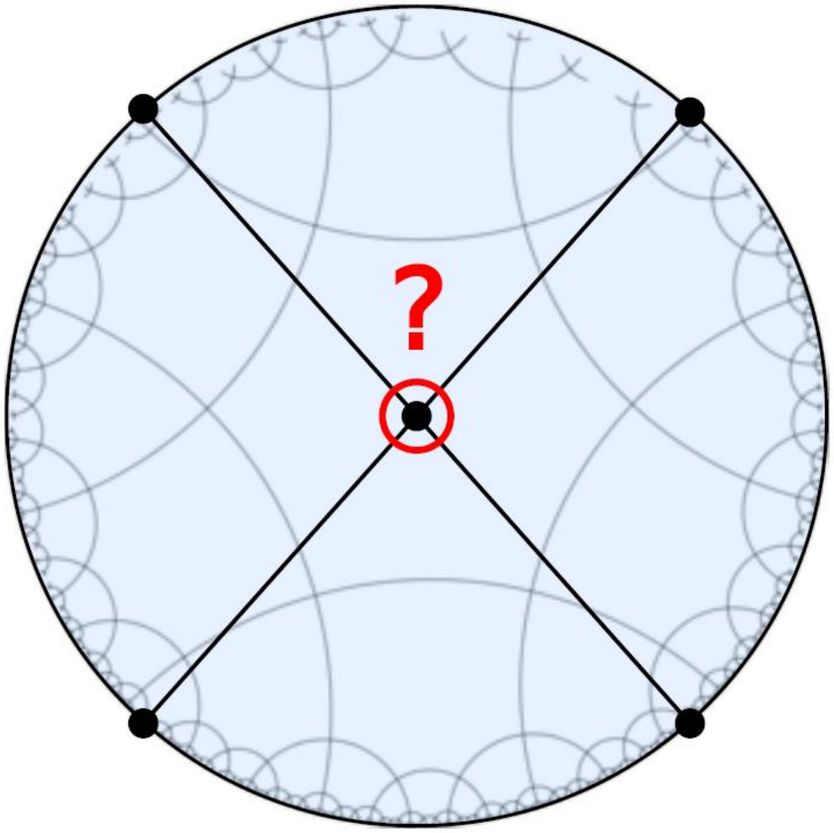
Cubic Interactions in higher-spin theory from CFT

Massimo Taronna

Based on: arXiv:1603.00022 & in preparation (with C. Sleight)

Can holography help us understand higher-spin Interactions?



What do we want to know?

- Vasiliev’s equations are formulated in terms of infinitely many **auxiliary fields**: can we make contact with the standard formulation (**Fronsdal**)?

- Gauge invariant metric-like cubic interactions classified, can we **fix** their coupling constants?

- What can we learn in this procedure? **Check** AdS/CFT dualities, quartic interactions, ...

Conventional Approach: Noether

Take as starting point the Fronsdal Lagrangian [Fronsdal ’78]

$$S^{(2)} = \sum_s \int \frac{1}{2}\, \varphi^{\mu_1\dots\mu_s}(\Box - m^2)\varphi_{\mu_1\dots\mu_s} + \dots$$

Consider a **weak field expansion** of a would be non-linear action and enforce gauge invariance:

$$\begin{aligned} S &= S^{(2)} + S^{(3)} + S^{(4)} + \dots \\ \delta\varphi &= \delta^{(0)}\varphi + \delta^{(1)}\varphi + \dots \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \delta^{(0)}S^{(2)} &= 0 \\ \delta^{(1)}S^{(2)} + \delta^{(0)}S^{(3)} &= 0 \\ \delta^{(2)}S^{(2)} + \delta^{(1)}S^{(3)} + \delta^{(0)}S^{(4)} &= 0 \end{aligned}$$

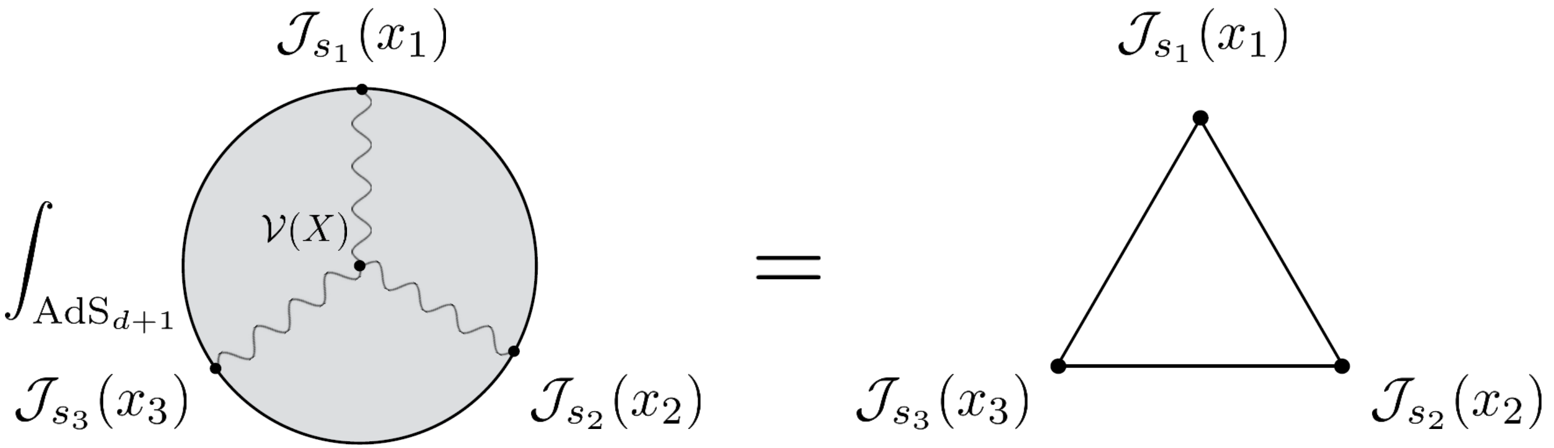
…

Becomes more and more **involved** beyond the cubic order

Holographic Approach

Higher-spin theory on AdS$_{d+1}$ ⟷ Free O(N) vector model

[Sezgin-Sundell, Klebanov-Polyakov, '02]



Solve the above equation for the bulk vertices $\mathcal{V}(X)$

Basic Idea

Dictionary:

Correlation Functions $\longleftrightarrow$ Witten Diagrams

Basic Idea

Dictionary:

Correlation Functions ⟷ Witten Diagrams

Simple in the free O(N) model

Ansatz for the bulk interactions

Basic Idea

Dictionary:

Correlation Functions $\longleftrightarrow$ Witten Diagrams

Simple in the free O(N) model

Ansatz for the bulk interactions

- Extract higher-spin interactions from free CFT correlators
- Ambient space formalism puts on equal footing bulk and boundary

Mellin amplitudes, bootstrap, ...

Outline

- **Lighting Review:** The Klebanov Polyakov Conjecture

- Holographic **reconstruction** of higher-spin cubic couplings

- Checks of the duality and open questions

Klebanov-Polyakov Conjecture

Klebanov-Polyakov Conjecture

Boundary: Free Scalar O(N) vector model in d dimensions (singlet sector)

$$S = \frac{1}{2}\sum_{a=1}^{N} \int \partial_i \phi^a \, \partial^i \phi^a$$

Single Trace sector:

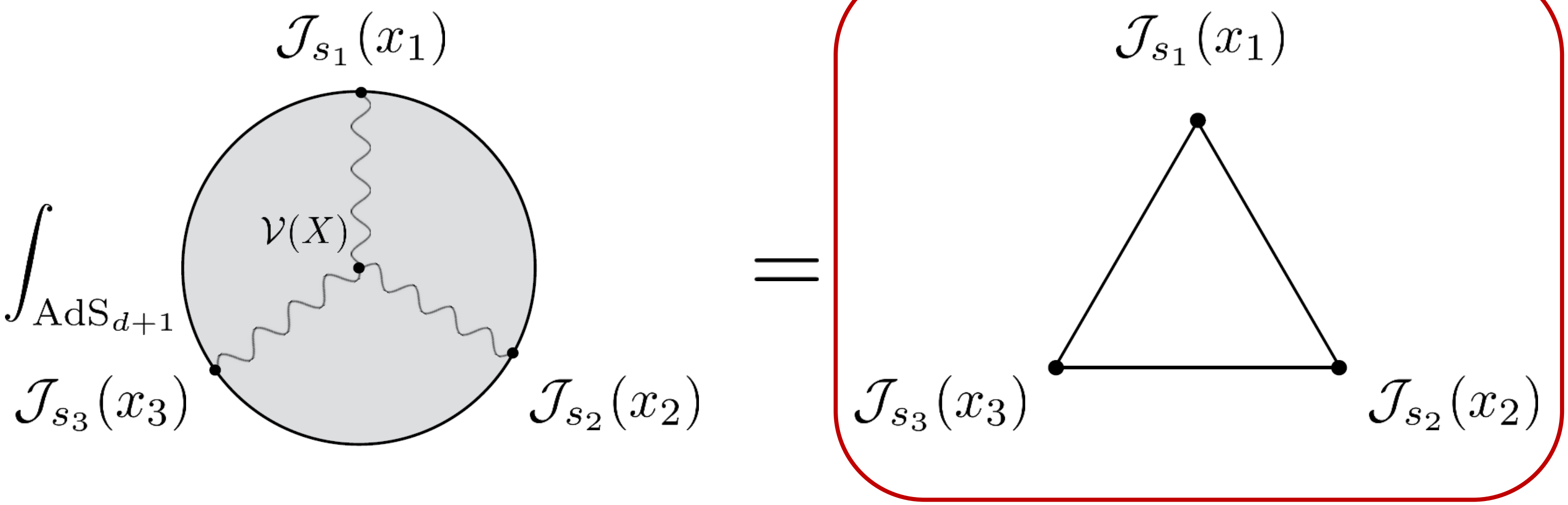
Scalar: $\mathcal{O} = \phi^a \phi^a$ $\Delta_0 = d - 2$

Conserved currents: $\mathcal{J}_{i_1 \ldots i_s} = \phi^a \partial_{(i_1} \ldots \partial_{i_s)} \phi^a + \ldots$ $\Delta_s = d - 2 + s$

(on-shell: $\partial^2 \phi^a \approx 0 \quad \Longrightarrow \quad \partial^{i_1} \mathcal{J}_{i_1 \ldots i_s} \approx 0$)

Dictionary: $\underset{\substack{\uparrow \\ \text{bulk scalar}}}{\varphi_0} \longleftrightarrow \mathcal{O}$ $\underset{\substack{\uparrow \\ \text{spin-s gauge field}}}{\varphi_{\mu_1 \ldots \mu_s}} \longleftrightarrow \mathcal{J}_{i_1 \ldots i_s}$

Holographic Reconstruction



Free Scalar OPE coefficients

Old trick to describe primary CFT operators [Craigie, Dobrev, Todorov ‘83]

$$\mathcal{J}_s\left(x|z\right) \equiv \mathcal{J}_{i_1\ldots i_s} z^{i_1} \ldots z^{i_s} = f^{(s)}\left(z\cdot\partial_{x_1}, z\cdot\partial_{x_2}\right) : \phi^a\left(x_1\right)\phi^a\left(x_2\right) : \Big|_{x_1,x_2\to x}$$

Conformal Boost generator

$$K^j J_{i_1\ldots i_s} = 0 \qquad \Longrightarrow \qquad f^{(s)}\left(x,y\right) = \left(x+y\right)^s\, C_s^{\left(\frac{\Delta-1}{2}\right)}\left(\frac{x-y}{x+y}\right)$$

Allows the seamless application of **Wick’s theorem**

Free Scalar OPE coefficients

Conformal invariance **fixes** 2pt and 3pt functions up to **coefficients**:

$$\langle \mathcal{J}_{s_1} \mathcal{J}_{s_2} \rangle = \mathsf{C}_{\mathcal{J}_{s_1}} \frac{\delta_{s_1,s_2}}{(x_{12}^2)^{\Delta}} \mathsf{H}_3^s$$

$$\langle \mathcal{J}_{s_1} \mathcal{J}_{s_2} \mathcal{J}_{s_3} \rangle = \sum_{n_i} \mathsf{C}_{s_1,s_2,s_3}^{n_1,n_2,n_3} \frac{\mathsf{Y}_1^{s_1-n_2-n_3} \mathsf{Y}_2^{s_2-n_3-n_1} \mathsf{Y}_3^{s_3-n_1-n_2} \mathsf{H}_1^{n_1} \mathsf{H}_2^{n_2} \mathsf{H}_3^{n_3}}{(x_{12}^2)^{\frac{\tau_1+\tau_2-\tau_3}{2}} (x_{23}^2)^{\frac{\tau_2+\tau_3-\tau_1}{2}} (x_{31}^2)^{\frac{\tau_3+\tau_1-\tau_2}{2}}}$$

6 conformal **structures** for 3pt:

$$x_{ij} \equiv x_i - x_j$$

$$\mathsf{Y}_1 = \frac{z_1 \cdot x_{12}}{x_{12}^2} - \dots \qquad \mathsf{H}_1 = \frac{1}{x_{23}^2} \left(z_2 \cdot z_3 + \dots \right)$$

Free Scalar OPE coefficients

Conformal invariance **fixes** 2pt and 3pt functions up to **coefficients**:

$$\langle \mathcal{J}_{s_1}\mathcal{J}_{s_2}\rangle = \boxed{\mathsf{C}_{\mathcal{J}_{s_1}}} \frac{\delta_{s_1,s_2}}{(x_{12}^2)^{\Delta}} \mathsf{H}_3^{s}$$

$$\langle \mathcal{J}_{s_1}\mathcal{J}_{s_2}\mathcal{J}_{s_3}\rangle = \sum_{n_i} \boxed{\mathsf{C}^{n_1,n_2,n_3}_{s_1,s_2,s_3}} \frac{\mathsf{Y}_1^{s_1-n_2-n_3}\mathsf{Y}_2^{s_2-n_3-n_1}\mathsf{Y}_3^{s_3-n_1-n_2}\mathsf{H}_1^{n_1}\mathsf{H}_2^{n_2}\mathsf{H}_3^{n_3}}{(x_{12}^2)^{\frac{\tau_1+\tau_2-\tau_3}{2}}(x_{23}^2)^{\frac{\tau_2+\tau_3-\tau_1}{2}}(x_{31}^2)^{\frac{\tau_3+\tau_1-\tau_2}{2}}}$$

6 conformal **structures** for 3pt:

$$x_{ij} \equiv x_i - x_j$$

$$\mathsf{Y}_1 = \frac{z_1 \cdot x_{12}}{x_{12}^2} - \ldots \qquad \mathsf{H}_1 = \frac{1}{x_{23}^2}\left(z_2 \cdot z_3 + \ldots\right)$$

OPE coefficients (invariant under rescaling of operators):

$$\left(\mathsf{c}^{n_1,n_2,n_3}_{s_1,s_2,s_3}\right)^2 \equiv \frac{\left(\mathsf{C}^{n_1,n_2,n_3}_{s_1,s_2,s_3}\right)^2}{\mathsf{C}_{\mathcal{J}_{s_1}}\mathsf{C}_{\mathcal{J}_{s_2}}\mathsf{C}_{\mathcal{J}_{s_3}}}$$

Free Scalar OPE coefficients

The final result **factorises** and reproduces conserved conformal structures

$$\Delta = d-2$$

$$\mathsf{C}^{0,0,0}_{s_1,s_2,s_3} = N\prod_{i=1}^{3}\mathsf{c}_{s_i}\,, \qquad \mathsf{c}^2_{s_i} = \frac{\sqrt{\pi}\,2^{-\Delta-s_i+3}\,\Gamma(s_i+\frac{\Delta}{2})\Gamma(s_i+\Delta-1)}{N\,s_i!\,\Gamma(s_i+\frac{\Delta-1}{2})\Gamma(\frac{\Delta}{2})^2}$$

00s OPE coeff.

$$\mathsf{C}^{n_1,n_2,n_3}_{s_1,s_2,s_3} = \frac{2^{-(n_1+n_2+n_3)}s_1!s_2!s_3!}{(s_1-n_2-n_3)!(s_2-n_3-n_1)!(s_3-n_1-n_2)!n_1!n_2!n_3!}\,\frac{\mathsf{C}^{0,0,0}_{s_1,s_2,s_3}}{(\frac{\Delta}{2})_{n_1}(\frac{\Delta}{2})_{n_2}(\frac{\Delta}{2})_{n_3}}$$

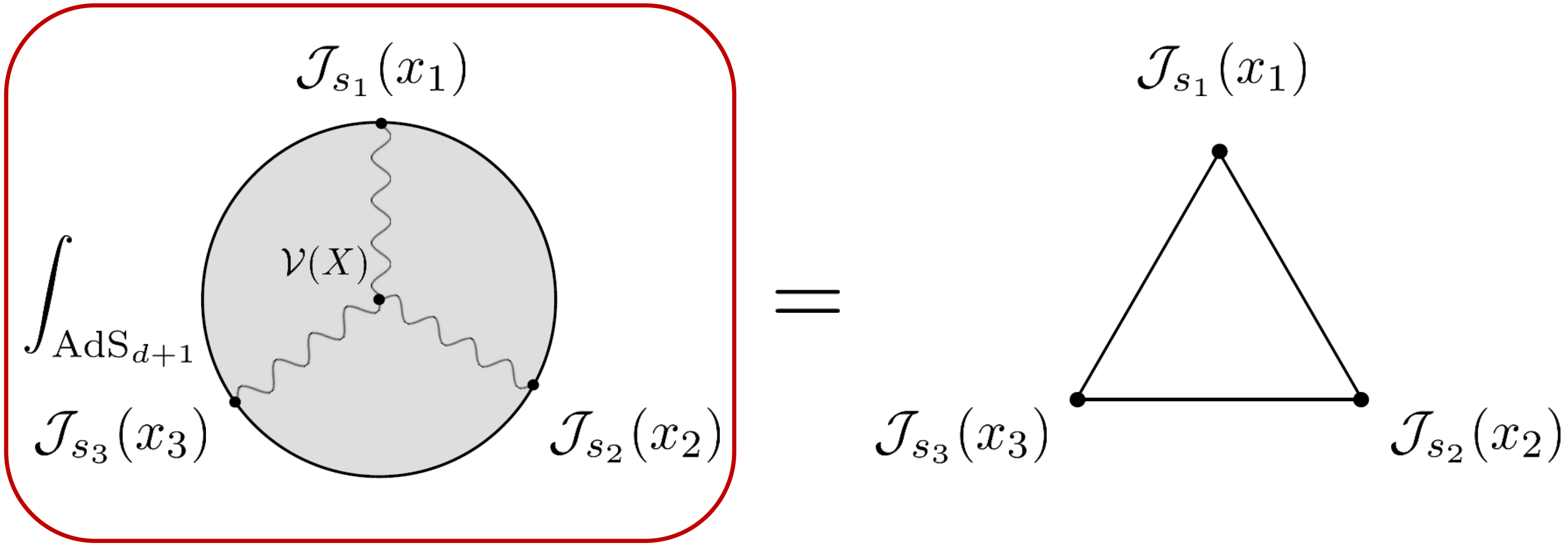
The full **N-point function** of arbitrary spin primary single trace operators can be resummed into a Bessel function

$$\langle \mathcal{J}_{s_1}(x_1|z_1)\mathcal{J}_{s_2}(x_2|z_2)\ldots\mathcal{J}_{s_n}(x_n|z_n)\rangle =$$

$$\frac{N}{(x^2_{12})^{\Delta/2}(x^2_{23})^{\Delta/2}\ldots(x^2_{n1})^{\Delta/2}}\left(\prod_{i=1}^{n}\mathsf{c}_{s_i}\,q_i^{\frac{1}{2}-\frac{\Delta}{4}}\Gamma(\tfrac{\Delta}{2})\,J_{\frac{\Delta-2}{2}}\left(\sqrt{q_i}\right)\right)\mathsf{Y}_1^{s_1}\mathsf{Y}_2^{s_2}\ldots\mathsf{Y}_n^{s_n} + \text{perm.}$$

$$q_i = H_i\partial_{Y_{i-1}}\partial_{Y_{i+1}}$$

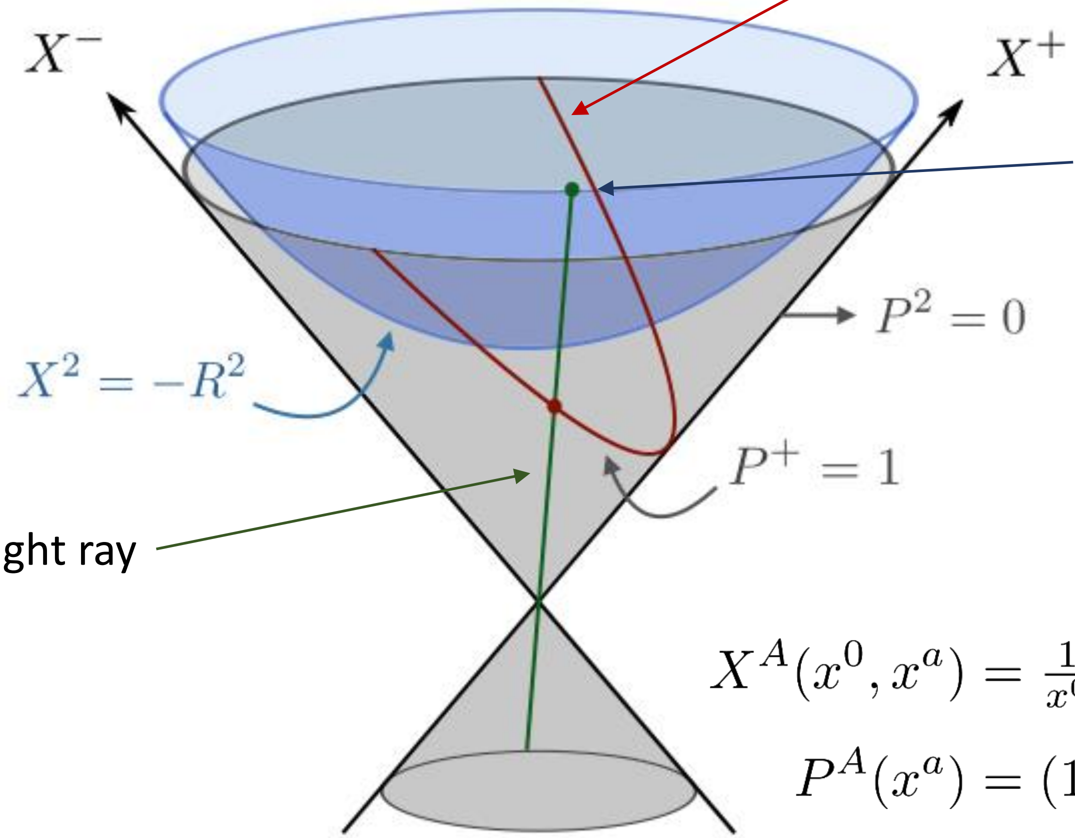
Holographic Reconstruction



Ambient Space Trick

[Dirac, Fronsdal, ...]

$$X^A = (X^+, X^-, X^a)$$



Poincaré Section

boundary pt.

light ray

$$X^A(x^0, x^a) = \frac{1}{x^0}(1, x_0^2 + x^2, x^a)$$

$$P^A(x^a) = (1, x^2, x^a)$$

$$\nabla^M = \partial_X^M - \frac{1}{X^2}(X^M X \cdot \partial_X + \dots)$$

Bulk Cubic Couplings

Most general bulk coupling: sum of the following **building blocks**:

$$I^{n_1,n_2,n_3}_{s_1,s_2,s_3}(\Phi_i) = \eta^{M_1(n_3)M_2(n_3)}\eta^{M_2(n_1)M_3(n_1)}\eta^{M_3(n_2)M_1(n_2)}(\partial^{N_3(k_3)}\Phi_{M_1(n_2+n_3)N_1(k_1)})$$
$$\times\ (\partial^{N_1(k_1)}\Phi_{M_2(n_3+n_1)N_2(k_2)})(\partial^{N_2(k_2)}\Phi_{M_3(n_1+n_2)N_3(k_3)})$$

The ansatz for the bulk vertex reads:

$$\mathcal{V} = \sum_{s_i,n_i} g^{n_1,n_2,n_3}_{s_1,s_2,s_3}\ I^{n_1,n_2,n_3}_{s_1,s_2,s_3}$$

Bulk Cubic Couplings

Most general bulk coupling: sum of the following **building blocks**:

$$I^{n_1,n_2,n_3}_{s_1,s_2,s_3}(\Phi_i) = \eta^{M_1(n_3)M_2(n_3)}\eta^{M_2(n_1)M_3(n_1)}\eta^{M_3(n_2)M_1(n_2)}(\partial^{N_3(k_3)}\Phi_{M_1(n_2+n_3)N_1(k_1)}) \\ \times (\partial^{N_1(k_1)}\Phi_{M_2(n_3+n_1)N_2(k_2)})(\partial^{N_2(k_2)}\Phi_{M_3(n_1+n_2)N_3(k_3)})$$

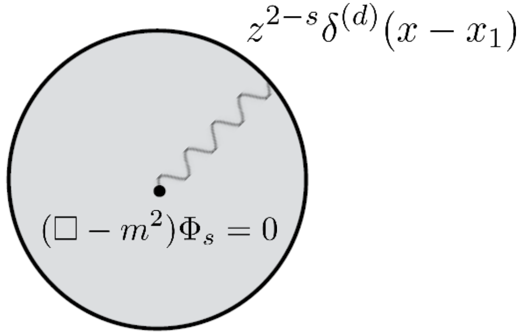
The ansatz for the bulk vertex reads:

$$\mathcal{V} = \sum_{s_i,n_i} g^{n_1,n_2,n_3}_{s_1,s_2,s_3} I^{n_1,n_2,n_3}_{s_1,s_2,s_3}$$

Need to solve for the relative coupling constants

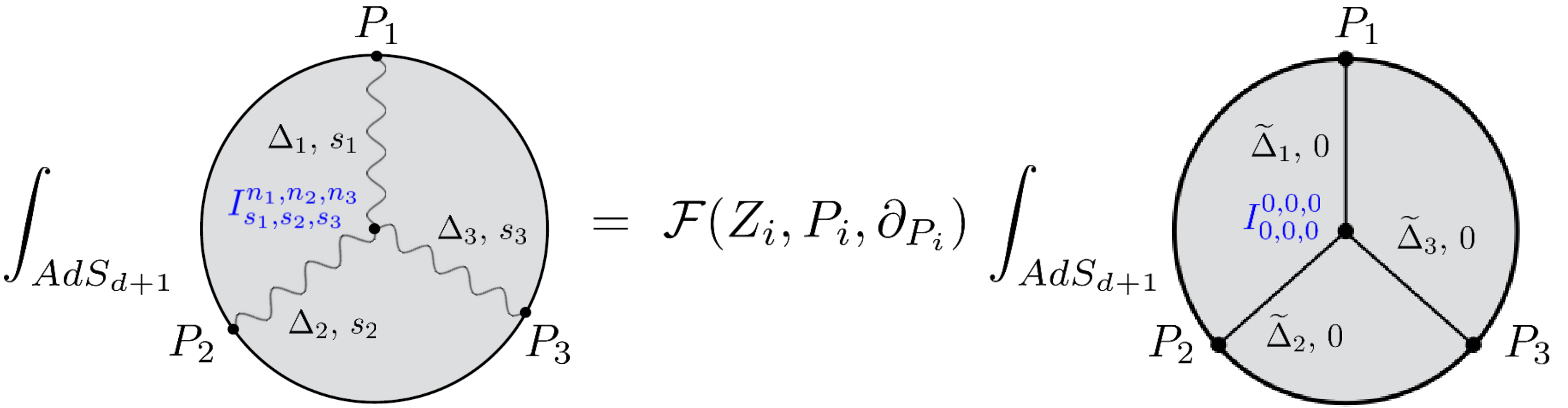
Plug boundary to bulk propagators and perform the **integral** over AdS:

$$\Phi_s \sim \frac{1}{(-2P(x)\cdot X)^{\Delta}}(\dots)$$



Integral over AdS

Trick: reduce the integral over AdS of a generic cubic coupling to its **scalar seed**



$$\widetilde{\Delta}_i = \Delta_i + s_i - n_{i-1} - n_{i+1}$$

Schwinger parameterised form of the propagator very useful:

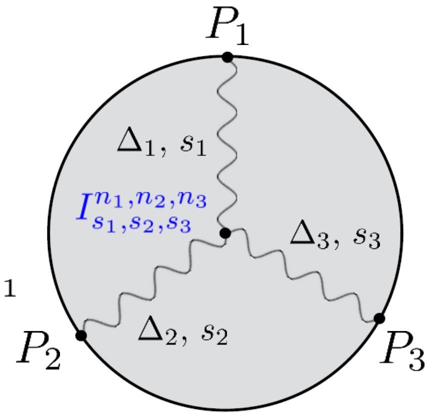
$$\int_{\mathrm{AdS}_{d+1}} dX \left(\prod_{i=1}^{3} \frac{dt_i}{t_i}\, t_i^{\widetilde{\Delta}_i} \right) e^{2(t_1 P_1 + t_2 P_2 + t_3 P_3)\cdot X}$$

$$= \pi^{\frac{d}{2}} \Gamma\left(\frac{\sum_{i=1}^{3} \widetilde{\Delta}_i - d}{2} \right) \int_0^\infty \prod_{i=1}^{3} \left(\frac{dt_i}{t_i} t_i^{\widetilde{\Delta}_i} \right) e^{2(t_1 t_2 P_1 \cdot P_2 + \text{cyclic})}$$

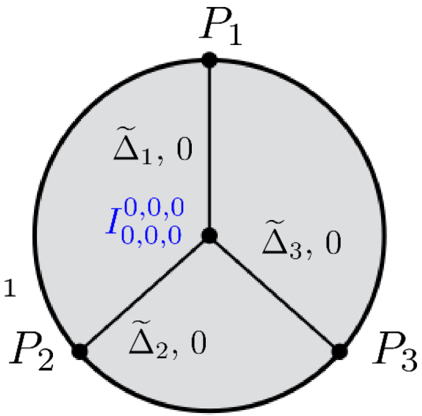
Integral over AdS

Trick: reduce the integral over AdS of a generic cubic coupling to its **scalar seed**

$$\int_{AdS_{d+1}}$$



$$= \mathcal{F}(Z_i, P_i, \partial_{P_i}) \int_{AdS_{d+1}}$$



$$\widetilde{\Delta}_i = \Delta_i + s_i - n_{i-1} - n_{i+1}$$

Schwinger parameterised form of the propagator very useful:

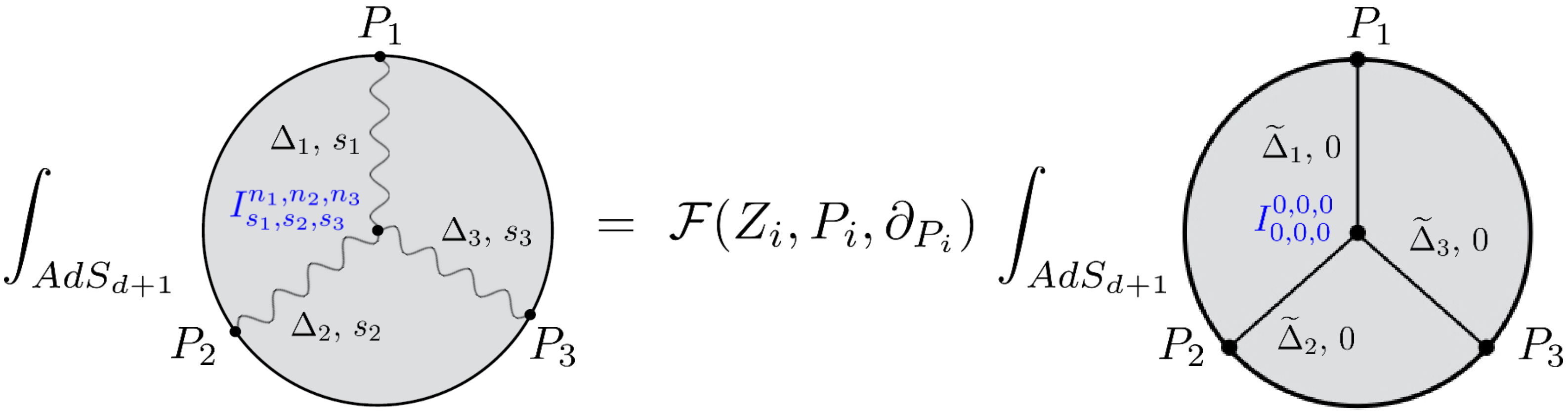
$$\int_{\mathrm{AdS}_{d+1}} dX \left(\prod_{i=1}^{3} \frac{dt_i}{t_i} t_i^{\widetilde{\Delta}_i}\right) e^{2(t_1P_1+t_2P_2+t_3P_3)\cdot X}$$

$$= \pi^{\frac{d}{2}} \boxed{\Gamma\left(\frac{\sum_{i=1}^{3} \widetilde{\Delta}_i - d}{2}\right)} \int_0^\infty \prod_{i=1}^{3} \left(\frac{dt_i}{t_i} t_i^{\widetilde{\Delta}_i}\right) e^{2(t_1t_2P_1\cdot P_2+\text{cyclic})}$$

Only non-factorised dependence on s_i

Integral over AdS

Trick: reduce the integral over AdS of a generic cubic coupling to its **scalar seed**



$$\widetilde{\Delta}_i = \Delta_i + s_i - n_{i-1} - n_{i+1}$$

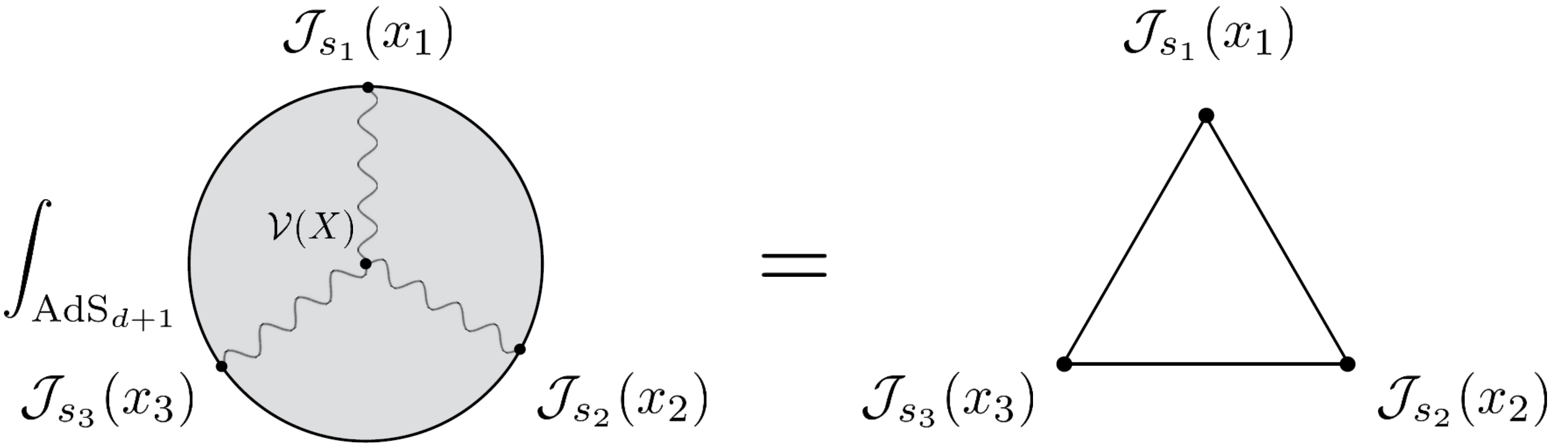
Schwinger parameterised form of the propagator very useful:

$$\int_{\mathrm{AdS}_{d+1}} dX \left(\prod_{i=1}^{3} \frac{dt_i}{t_i} t_i^{\widetilde{\Delta}_i}\right) e^{2(t_1P_1+t_2P_2+t_3P_3)\cdot X}$$

$$= \pi^{\frac{d}{2}}\Gamma\left(\frac{\sum_{i=1}^{3} \widetilde{\Delta}_i - d}{2}\right) \boxed{\int_0^\infty \prod_{i=1}^{3} \left(\frac{dt_i}{t_i} t_i^{\widetilde{\Delta}_i}\right) e^{2(t_1t_2P_1\cdot P_2+\text{cyclic})}}$$

Gives Mellin form of 3 pt correlator via Symanzick star formula

Complete Higher-Spin Cubic Action



$$\mathcal{V} = \sum_{s_1,s_2,s_3} g_{s_1,s_2,s_3} I^{0,0,0}_{s_1,s_2,s_3} \qquad g_{s_1,s_2,s_3} = \frac{1}{\sqrt{N}} \frac{\pi^{\frac{d-3}{4}} 2^{\frac{3d-1+s_1+s_2+s_3}{2}}}{\Gamma(d+s_1+s_2+s_3-3)} \prod_{i=1}^{3} \sqrt{\frac{\Gamma(s_i+\frac{d-1}{2})}{\Gamma\left(s_i+1\right)}}$$

We obtain the **complete higher-spin cubic action**

$$I^{0,0,0}_{s_1,s_2,s_3}(\Phi_i) = (\partial^{N_3(k_3)}\Phi_{N_1(k_1)})(\partial^{N_1(k_1)}\Phi_{N_2(k_2)})(\partial^{N_2(k_2)}\Phi_{N_3(k_3)})$$

Radial Reduction

The ambient space form of the coupling looks $s_1+s_2+s_3$ derivative but one can reinstate the AdS covariant derivatives:

$$\widetilde{\mathcal{Y}}_1 = \partial_{U_1} \cdot \nabla_2 \qquad \widetilde{\mathcal{Y}}_2 = \partial_{U_2} \cdot \nabla_3 \qquad \widetilde{\mathcal{Y}}_3 = \partial_{U_3} \cdot \nabla_1$$

The problem can be solved by a recursion relation:

$$(\ldots)\, \partial_X^l \nabla^k = (\ldots)\, \partial_X^{l-1} \nabla^{k+1} + \text{lower derivative terms}$$

In the 1-1-1 case for instance the YM vertex is recovered:

$$\mathcal{Y}_1 \mathcal{Y}_2 \mathcal{Y}_3 \sim F^3 + (d-1)[A_\mu, A_\nu] F^{\mu\nu}$$

Recall: $\mathcal{Y}_1 = \partial_{U_1} \cdot \partial_{X_2}$

Radial Reduction

The ambient space form of the coupling looks $s_1+s_2+s_3$ derivative but one can reinstate the AdS covariant derivatives:

$$\widetilde{\mathcal{Y}}_1 = \partial_{U_1} \cdot \nabla_2 \qquad \widetilde{\mathcal{Y}}_2 = \partial_{U_2} \cdot \nabla_3 \qquad \widetilde{\mathcal{Y}}_3 = \partial_{U_3} \cdot \nabla_1$$

The problem can be solved by a recursion relation:

$$(\ldots)\, \partial_X^l \nabla^k = (\ldots)\, \partial_X^{l-1} \nabla^{k+1} + \text{lower derivative terms}$$

In the 1-1-1 case for instance the YM vertex is recovered:

$$\mathcal{Y}_1 \mathcal{Y}_2 \mathcal{Y}_3 \sim F^3 + \boxed{(d-1)[A_\mu, A_\nu]F^{\mu\nu}}$$

YM

Recall: $\mathcal{Y}_1 = \partial_{U_1} \cdot \partial_{X_2}$

Some Checks

The Holographically Reconstructed HS algebra

Cubic couplings induce deformations of gauge transformations and gauge symmetries

$$\int \left[(\delta^{(1)}\Phi)\Box\Phi + \delta^{(0)}\mathcal{V} \right] = 0$$

Solve for the induced gauge transformations

$$\delta^{(0)}_{[\epsilon_1}\delta^{(1)}_{\epsilon_2]} \approx \delta^{(0)}_{[\![\epsilon_1,\epsilon_2]\!]^{(0)}}$$

Solve for the induced bracket

At cubic order no condition is imposed on the deformations but at quartic:

Jacobi:

$$[\![\epsilon_1, [\![\epsilon_2,\epsilon_3]\!]^{(0)}\,]\!]^{(0)} + \text{cyclic} = 0$$

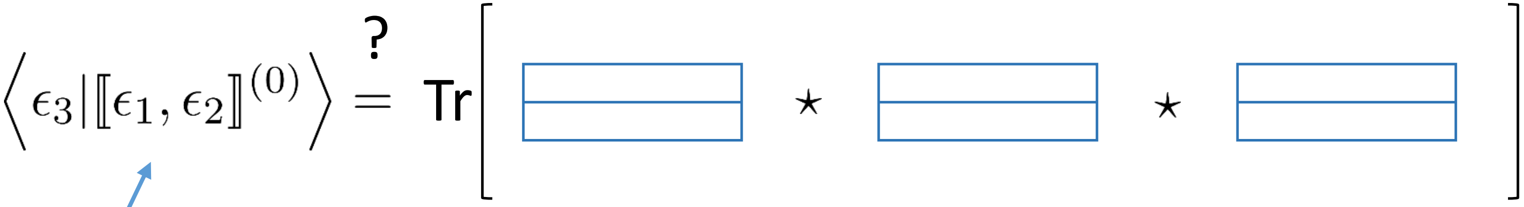
Admissibility:

$$\delta^{(1)}_{[\epsilon_1}\delta^{(1)}_{\epsilon_2]} \approx \delta^{(1)}_{[\![\epsilon_1,\epsilon_2]\!]^{(0)}}$$

$$\left(\nabla_\mu \epsilon_{\mu(s-1)} = 0\right)$$

The Holographically Reconstructed HS algebra

The deformation of the gauge algebra induced by the cubic couplings **matches** the structure constants of the HS algebras **in any D**



$$\left\langle \epsilon_3 \middle| [\![\epsilon_1, \epsilon_2]\!]^{(0)} \right\rangle \overset{?}{=} \mathrm{Tr}\left[\quad \star \quad \star \quad \right]$$

[Eastwood, Vasiliev; Joung, Mkrtchyan ...]

The **reconstructed bracket** reproduces as expected the HS algebra structure constants with the following normalisation of the invariant bilinear:

$$\mathrm{Tr}(T_s \star T_s) = \frac{1}{(s-1)^2} \frac{\pi^{\frac{d}{2}-1} s\, 2^{d-4s+7} \Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-3}{2}\right) \Gamma\left(\frac{d-5}{2}+s\right)}$$

[C.Sleight & M.T. in preparation]

How can we test the Born-Infeld couplings?

So far **most** of the tests of HS holography rely on the Jacobi & Admissibility conditions

What about Born Infeld couplings which are blind to these tests??

Metsaev fixed **all cubic coupling** in flat space by requiring **Poincaré invariance** in the flat limit up to the quartic order:

$$\mathcal{V} = \sum_{|s_i|=0}^{\infty} \frac{(il)^{s_1+s_2+s_3}}{\Gamma(s_1+s_2+s_3)} \left[\partial_{x_1}(\partial_2^+ - \partial_3^+) + \text{cyclic}\right]^{s_1+s_2+s_3} \; \frac{\varphi_{s_1}}{(\partial_{x_1}^+)^{s_1}} \frac{\varphi_{s_2}}{(\partial_{x_2}^+)^{s_2}} \frac{\varphi_{s_3}}{(\partial_{x_3}^+)^{s_3}} + h.c.$$

In the flat limit the **highest-derivative** part (Born-Infeld, +++, ---) of the holographically reconstructed couplings **match** those found by Metsaev in ’93!

Open Question: how do we test Born-Infeld couplings in AdS in general?

Summary

- Holographic reconstruction allows to fix the complete cubic action of higher-spin theories in AdS
- The coupling reconstructed are not only gauge invariant but solve the Noether procedure up to the quartic order (first test of the duality in d>4)
- Completion to the de Donder gauge form obtained

$$\mathcal{V} = \sum_{s_1,s_2,s_3} g_{s_1,s_2,s_3} I^{0,0,0}_{s_1,s_2,s_3}$$

$$g_{s_1,s_2,s_3} = \frac{1}{\sqrt{N}} \frac{\pi^{\frac{d-3}{4}} 2^{\frac{3d-1+s_1+s_2+s_3}{2}}}{\Gamma(d+s_1+s_2+s_3-3)} \prod_{i=1}^{3} \sqrt{\frac{\Gamma(s_i+\frac{d-1}{2})}{\Gamma(s_i+1)}}$$

Outlook

- What about other free CFTs? (free-fermion, ...)
- There are also parity violating structures in 3d dual to parity violating HS theories in the 4d bulk
- Quartic vertex? Loops?

$$\mathcal{V} = \sum_{s_1,s_2,s_3} g_{s_1,s_2,s_3} I^{0,0,0}_{s_1,s_2,s_3}$$

$$g_{s_1,s_2,s_3} = \frac{1}{\sqrt{N}} \frac{\pi^{\frac{d-3}{4}} 2^{\frac{3d-1+s_1+s_2+s_3}{2}}}{\Gamma(d+s_1+s_2+s_3-3)} \prod_{i=1}^{3} \sqrt{\frac{\Gamma(s_i+\frac{d-1}{2})}{\Gamma\left(s_i+1\right)}}$$