Conformal blocks and geodesic networks

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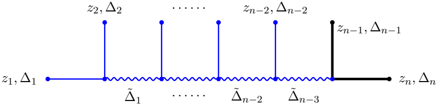
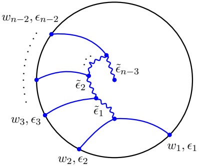
The duality statement





$$f_1(z|\epsilon, \tilde{\epsilon}) = S(w|\epsilon, \tilde{\epsilon})$$

Outline

- Classical conformal blocks

Zamolodchikov 1994
Fitzpatrick, Kaplan, Walters 2014
Alkalaev, Belavin 2015
Banerjee, Datta, Sinha 2016

- Geodesic approximation

Hijano, Kraus, Snively 2015
Alkalaev, Belavin 2015

- Bulk/boundary correspondence in the n-point case

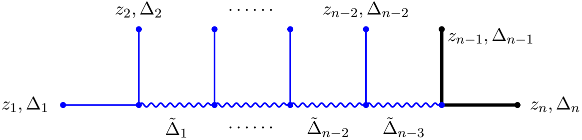
- Conclusions and outlooks

Classical Virasoro conformal block

The n-point correlation function of $V_{\Delta_i}(z_i)$, $i = 1, ..., n$ can be decomposed into conformal blocks

$$\mathcal{F}(z_1, ..., z_n | \Delta_1, ..., \Delta_n; \tilde{\Delta}_1, ..., \tilde{\Delta}_{n-3} | c)$$

which are conveniently depicted as



In the semiclassical limit $c \to \infty$ the conformal blocks exponentiate as

$$\mathcal{F}(z_i, \Delta_i, \tilde{\Delta}_j) = \exp\big[- c\, f(z_i, \epsilon_i, \tilde{\epsilon}_j)\big]$$

where $\epsilon_k = \frac{\Delta_k}{c}$ and $\tilde{\epsilon}_k = \frac{\tilde{\Delta}_k}{c}$ are *classical dimensions* and $f(z|\epsilon, \tilde{\epsilon})$ is the *classical* conformal block.

Auxiliary Fuchsian equation

We consider $(n+1)$-point correlation functions with one degenerate operator. The singular vector decoupling condition

$$\left[c\frac{\partial^2}{\partial y^2}+\sum_{i=1}^{5}\left(\frac{\Delta_i}{(y-z_i)^2}+\frac{1}{y-z_i}\frac{\partial}{\partial z_i}\right)\right]\langle V_{12}(y)V_1(z_1)\cdots V_n(z_n)\rangle=0\,.$$

In the classical limit $c\to\infty$ the $(n+1)$-point auxiliary correlation function behaves as

$$\langle V_{12}(y)V_1(z_1)\cdots V_n(z_n)\rangle\Big|_{c\to\infty}\;\approx\;\psi(y|z)\exp\left[-c\,f(z_i,\epsilon_i,\tilde{\epsilon}_j)\right]\,,$$

where $f(z_i,\epsilon_i,\tilde{\epsilon}_j)$ is the classical block and $\psi(y|z)$ is governed by the Fuchsian equation

$$\frac{d^2\psi(y|z)}{dy^2}+T(y|z)\psi(y|z)=0\,,\qquad T(y|z)=\sum_{i=1}^{n}\left(\frac{\epsilon_i}{(y-z_i)^2}+\frac{c_i}{y-z_i}\right)$$

Here $T(z)$ is the stress-energy tensor and c_i are the accessory parameters

$$c_i(z)=\frac{\partial f(z)}{\partial z_i}\,,\qquad i=1,...,n$$

The asymptotic behavior $T(z)\sim z^{-4}$ at infinity yields the constraints

$$\sum_{i=1}^{n}c_i=0\,,\qquad \sum_{i=1}^{n}(c_iz_i+\epsilon_i)=0\,,\qquad \sum_{i=1}^{n}(c_iz_i^2+2\epsilon_iz_i)=0$$

There are $n-3$ independent accessory parameters, $c_2,\ldots,c_{n-2}$.

Heavy-light approximation. We consider the case of *two* background operators. Let $\epsilon_{n-1} = \epsilon_n \equiv \epsilon_h$ be the background heavy dimension, while ϵ_i, $i = 1, \dots, n-2$ be perturbative heavy dimensions,

$$\epsilon_i / \epsilon_h \ll 1$$

Then, the Fuchsian equation can be solved perturbatively. We expand all functions as

$$\psi(y,z) = \psi^{(0)}(y,z) + \psi^{(1)}(y,z) + \dots$$

$$T(y,z) = T^{(0)}(y,z) + T^{(1)}(y,z) + \dots$$

$$c_i(z) = c_i^{(0)}(z) + c_i^{(1)}(z) + \dots$$

$$f(z) = f^{(0)}(z) + f^{(1)}(z) + f^{(2)}(z) + \dots$$

where expansion parameters are perturbative heavy dimensions ϵ_i.

Accessory parameter equations. Using the monodromy method we find the constraints

$$\left(I_{++}^{(k)}\right)^2 + I_{-+}^{(k)}\, I_{+-}^{(k)} = -4\pi^2 \tilde{\epsilon}_k^2\,, \qquad k = 1, \dots, n-3$$

where quantities $I_{\pm\pm}^{(k)}$ are linear in the accessory parameters, e.g.,

$$I_{+-}^{(k)} \;\sim\; \alpha\epsilon_1 + \sum_{i=2}^{n-2}(c_i(1-z_i) - \epsilon_i) - \sum_{i=2}^{k+1}(1-z_i)^{\alpha}(c_i(1-z_i) - \epsilon_i(1+\alpha))$$

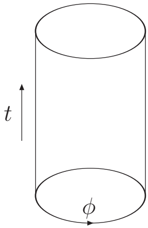
Here $\alpha = \sqrt{1 - 4\epsilon_h}$ parameterizes the background dimension.

Dual picture

The heavy operators with equal conformal dimensions $\epsilon_n = \epsilon_{n-1} \equiv \epsilon_h$ produce an asymptotically AdS_3 geometry identified either with an angular deficit or BTZ black hole geometry parameterized by

$$\alpha = \sqrt{1 - 4\epsilon_h}$$

- $\alpha^2 > 0$ for the conical singularity
- $\alpha^2 < 0$ for the BTZ black hole



The metric of the conical singularity reads

$$ds^2 = \frac{\alpha^2}{\cos^2 \rho}\Big(- dt^2 + \sin^2 \rho d\phi^2 + \frac{1}{\alpha^2} d\rho^2 \Big)$$

The perturbative heavy operators are realized via particular network of worldlines of $n-3$ classical point probes propagating in the background geometry formed by the two background heavy operators. Points w_i are boundary attachments of the perturbative operators.

The worldline action of a single massive particle with $m \sim \epsilon$ is

$$S = \epsilon \int_{\lambda'}^{\lambda''} d\lambda \sqrt{g_{tt}\dot{t}^2 + g_{\phi\phi}\dot{\phi}^2 + g_{\rho\rho}\dot{\rho}^2}$$

A geodesic is characterized by the endpoint coordinates and the angular momentum $s = \frac{|p_\phi|}{\alpha}$. Its length is given by the on-shell value S.

Most importantly, the time slice is the Poincare disk!

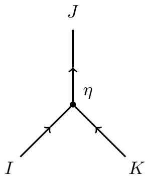
Cubic vertex on the disk

The vertex action for three distinct lines has the form

$$S_\star = \epsilon_I \int_{\circ I}^{\bullet} d\lambda\, L_I + \epsilon_J \int_{\circ J}^{\bullet} d\lambda\, L_J + \epsilon_K \int_{\circ K}^{\bullet} d\lambda\, L_K \;, \qquad I \neq J \neq K \;,$$

with the vertex point $\bullet$ and outer endpoints $\circ_A$, where $A = I, J, K$.
The equilibrium condition at the vertex point is given by

$$P^{(I)} + P^{(J)} + P^{(K)} = 0 \;,$$



where $P_m^{(A)} = \partial L_A / \partial \dot{X}^m_{(A)}$ are canonical momenta of three particles with coordinates $X^m_{(A)}$, where $m = \rho, \phi$ and $A = I, J, K$.
Recalling that $P_\phi^{(A)} = \pm \alpha s_A$, where the overall sign depends on the direction of the flow, we find that the radial and angular projections of the equilibrium condition are given by

$$\epsilon_I \sqrt{1 - s_I^2 \eta} - \epsilon_J \sqrt{1 - s_J^2 \eta} + \epsilon_K \sqrt{1 - s_K^2 \eta} = 0$$

$$\epsilon_I s_I + \epsilon_J s_J - \epsilon_K s_K = 0$$

where $\eta = \cot^2 \rho_{vert}$. The solution reads

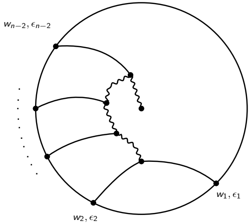
$$\eta = \frac{1 - \sigma_{IJ}^2}{s_I^2 + s_J^2 - 2\sigma_{IJ} s_I s_J} \;, \qquad \text{where} \qquad \sigma_{IJ} = \frac{\epsilon_I^2 + \epsilon_J^2 - \epsilon_K^2}{2\epsilon_I \epsilon_J}$$

Dual network

The total worldline action is given by the sum of $n-3$ vertex actions, *i.e.*,

$$S = \sum_{m=1}^{n-3} S_\star^{(m)}$$

where endpoints are connected to each other to form the network shown on the figure.



Given that the action functional S is stationary we find the equilibrium conditions at each vertex point

$$P_{(i+1)} + P_{(\tilde{i})} + P_{(\widetilde{i-1})} = 0\ , \qquad i = 1, \dots, n-3\ ,$$

and out-flowing momenta in all attachment points on the boundary and at the center of the disk,

$$P_{(A)} = \frac{\partial S}{\partial X_{(A)}}\ , \qquad A = 1, \dots, n-3, \widetilde{n-3}\ ,$$

where the last equality is assumed to be weak, *i.e.* the action S is evaluated on-shell. One can show that

$$s_k = \frac{1}{\alpha \epsilon_k} \frac{\partial S}{\partial w_k}$$

where w_k are boundary attachment coordinates.

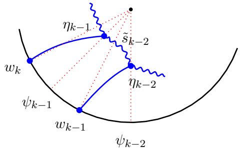
Angle separations

Angular separation of the geodesic segment with two endpoints having radial and angular positions $(\phi^{'}, \eta^{'})$ and $(\phi^{''}, \eta^{''})$ characterized by the angular parameter s can be represented as

$$i\alpha(\phi^{''} - \phi^{'}) = \ln \frac{\sqrt{1 - s^2\eta^{''}} - is\sqrt{1 + \eta^{''}}}{\sqrt{1 - s^2\eta^{'}} - is\sqrt{1 + \eta^{'}}}$$

The angular positions satisfy the balance equation

$$(w_k - w_{k-1}) + \Delta\phi_{k-1} = \Delta\phi_k + \Delta\tilde{\phi}_{k-2}$$



All in all, we obtain the system of radical equations

$$e^{i\alpha(w_k - w_{k-1})} \frac{1 - is_k}{1 - is_{k-1}} \frac{D^-_{k-1}}{D^+_k} = 1$$

where we introduced notation

$$D^-_k = (\sqrt{1 - s_k^2\eta_{k-1}} - is_k\sqrt{1 + \eta_{k-1}})(\sqrt{1 - \tilde{s}_{k-1}^2\eta_{k-1}} - i\tilde{s}_{k-1}\sqrt{1 + \eta_{k-1}})$$

$$D^+_k = (\sqrt{1 - s_k^2\eta_{k-1}} - is_k\sqrt{1 + \eta_{k-1}})(\sqrt{1 - \tilde{s}_{k-2}^2\eta_{k-1}} - i\tilde{s}_{k-2}\sqrt{1 + \eta_{k-1}})$$

Two systems

The conformal block and the mechanical action (geodesic length) read

$$c_k = \frac{\partial f}{\partial z_k} \qquad\qquad s_k = \frac{1}{\alpha \epsilon_k} \frac{\partial S}{\partial w_k}$$

On the boundary. The accessory equations

$$c_1 = -\sum_{i=2}^{n-2} \left[c_i (1 - z_i) - \epsilon_i \right] + \epsilon_1 \qquad\qquad (I_{++}^{(k)})^2 + I_{-+}^{(k)} I_{+-}^{(k)} = 4\pi^2\, \tilde{\epsilon}_k^2$$

where independent variables are $c_1, \ldots, c_{n-2}$. There are $(n-2)$ equations for $(n-2)$ variables.
In the bulk. The momentum equations

$$\epsilon_i s_i + \tilde{\epsilon}_{i-1} \tilde{s}_{i-1} - \tilde{\epsilon}_{i-2} \tilde{s}_{i-2} = 0$$

$$\tilde{\epsilon}_{k-1} \sqrt{1 - \tilde{s}_{k-1}^2 \eta_{k-1}} - \tilde{\epsilon}_{k-2} \sqrt{1 - \tilde{s}_{k-2}^2 \eta_{k-1}} - \epsilon_k \sqrt{1 - \tilde{s}_k^2 \eta_{k-1}} = 0$$

$$e^{i\alpha(w_k - w_{k-1})} \frac{1 - i s_k}{1 - i s_{k-1}} \frac{D_{k-1}^-}{D_k^+} = 1$$

where independent variables are angular parameters $s_1, \ldots, s_{n-2}$, $\tilde{s}_1, \ldots, \tilde{s}_{n-3}$, and vertex positions $\eta_1, \ldots, \eta_{n-3}$. There are $(3n-8)$ equations for $(3n-8)$ variables.

Weak equivalence

On the formal level, the problem is as follows. We consider a potential vector field

$$A_i(x) = \frac{\partial U(x)}{\partial x^i}$$

and impose the algebraic constraints

$$C^{(N)}_{\alpha}(A, B) = 0 \ , \qquad \alpha = 1, \dots, N \ ,$$

where B_k are possible auxiliary variables, $B = B(A)$. We consider two potential vector field systems defined by two different sets

$$\{x, U(x), A(x), B(x), C^{(N)}\} \ , \qquad \{y, \widetilde{U}(y), \widetilde{A}(y), \widetilde{B}(y), \widetilde{C}^{(\widetilde{N})}\}$$

Two systems are weakly equivalent if

$$C^{(N)}_{\alpha}(A, B) = 0 \ , \qquad C^{(\tilde{N})}_{\tilde{\beta}}(\widetilde{A}, \widetilde{B}) = 0 \ ,$$

have at least one common root $\{A^0_i(x)\} \to \{\widetilde{A}^0_i(y)\}$ under transformations

$$x \to y \ , \qquad U(x) \to \widetilde{U}(y)$$

In our case, the boundary system has no B-type variables which are characteristic of the bulk system. This is quite natural from the AdS/CFT perspective in the sense that not all bulk degrees of freedom are fundamental. Integrating out the local degrees of freedom identified here with B-type variables we are left with A-type variables which are fundamental boundary variables.

The duality statement

The conformal map (cylinder $\leftrightarrow$ plane) is given by

$$w = i \ln(1 - z)$$

The correspondence between a CFT with two background operators in the large central charge regime and dual geodesic networks on the conical singularity background claims that the perturbative classical n-point block and the on-shell worldline action of the dual network be related to each other as

$$f(z) = S(w) + i \sum_{k=1}^{n-2} \epsilon_k w_k$$

The accessory and angular parameters are related as

$$c_k = \epsilon_k \, \frac{1 \pm i\alpha s_k}{1 - z_k}$$

with the convention that "$-$" at $k = 1$ and "$+$" at $k \neq 1$. In this case the bulk/boundary systems are weakly equivalent.

Conclusions & outlooks

Conclusions

- Our main result: n-point classical conformal blocks in the heavy-light approximation are equal (modulo the conformal map) to the lengths of dual geodesic networks for any n.
- The duality is shown without knowing explicit expressions of quantities on the both sides. We reformulated both bulk/boundary systems as the potential vector field equations, where vector components are subjected to the algebraic equations (weak equivalence).

Outlooks

- A possible future direction is to apply our technique to semiclassical CFTs on higher genius Riemann surfaces. In the torus case, *one-point* classical blocks were studied in

  M.Piatek, 2013

  while their holographic interpretation was proposed in

  K.Alkalaev and V.Belavin, 2016
- The semiclassical correspondence considered along these lines can be extended by including $1/c$ corrections. The 4-point case was studied in

  M.Beccaria, A.Fachechi, G.Macorini, 2015

  A.Fitzpatrick, J.Kaplan, 2016

  It would be interesting to understand how our results for n-point blocks connect with going beyond the leading $1/c$ order.