Perturbative analysis in higher-spin theories: physical sector

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Outline

- Vasiliev equations
- Homotopy trick
- Twisted derivative
- Adjoint derivative

Vasiliev equations

- Nonlinear HS equations for $W\left(x|Z^A, Y^A|k, \bar{k}|dx, \theta^A\right)$ and $B\left(x|Z^A, Y^A|k, \bar{k}\right)$:

$$\mathrm{d}_X W + W * W = -i\theta_\alpha \wedge \theta^\alpha \left(1 + \eta B * \varkappa k\right) - i\bar{\theta}_{\dot{\alpha}} \wedge \bar{\theta}^{\dot{\alpha}} \left(1 + \bar{\eta} B * \bar{\varkappa} \bar{k}\right),$$

$$\mathrm{d}_X B + W * B - B * W = 0.$$

- Evolution on auxiliary variables Z, θ encodes infinite number of HS vertices. Resolving it, one will get equations in physical sector

$$\mathrm{d}_X \omega = \mathcal{V}\left(\omega, \omega\right) + \mathcal{V}\left(\omega, \omega, C\right) + \mathcal{V}\left(\omega, \omega, C, C\right) + \ldots$$

$$\mathrm{d}_X C = \mathcal{V}\left(\omega, C\right) + \mathcal{V}\left(\omega, C, C\right) + \mathcal{V}\left(\omega, C, C, C\right) + \ldots$$

where $\omega\left(Y\right) = W\left(Z = 0, \theta = 0\right)$ describes all gauge-noninvariant d.o.f. of HS multiplet, while $C\left(Y\right) = B\left(Z = 0\right)$ – all gauge-invariant ones.

Perturbative analysis

- AdS_4 vacuum solution is provided by

$$B_0 = 0, \qquad W_0 = \phi_{AdS} + Z_A \theta^A,$$

where ϕ_{AdS} - space-time 1-form of AdS_4-connection

$$\phi_{AdS} = -\frac{i}{4}\phi^{AB} Y_A Y_B = -\frac{i}{4}\left(\omega^{AB} + h^{AB}\right) Y_A Y_B,$$

- While expanding Vasiliev equations over this vacuum two types of perturbative equations arise, in the adjoint and twisted adjoint sectors,

$$\Delta_{ad} f := -2i\mathrm{d}_Z f + \mathcal{D}_{ad} f = J,$$

$$\Delta_{tw} f := -2i\mathrm{d}_Z f + \mathcal{D}_{tw} f = J,$$

where

$$\mathrm{d}_Z := \theta^A \frac{\partial}{\partial Z^A},$$

$$\mathcal{D}_{ad} := \mathrm{d}_X + [\phi_{AdS}, \bullet]_* , \qquad \mathcal{D}_{tw} := \mathrm{d}_X - \frac{i}{4}\left[\omega^{AB} Y_A Y_B, \bullet\right]_* - \frac{i}{4}\left\{h^{AB} Y_A Y_B, \bullet\right\}_*$$

.

Homotopy trick

- Consider some nilpotent operator d, $\mathrm{d}^2 = 0$. If there is a homotopy operator ∂, $\partial^2 = 0$, such that

$$A := \{\mathrm{d}\,, \partial\}$$

is diagonalizable, then

$$H(\mathrm{d}) \subset KerA.$$

- This allows to write down a resolution of identity

$$\{\mathrm{d}\,, \mathrm{d}^*\} + \hat{h} = Id\,.$$

where $\hat{h}$ is a projector to $KerA$ and

$$\mathrm{d}^* := \partial A^*,$$

$$A^* A = AA^* = Id - \hat{h}$$

- Then

$$\mathrm{d}f = J \quad \Longrightarrow \quad f = \mathrm{d}^* J + \mathrm{d}\epsilon + g,$$

where $g \in H(\mathrm{d})$.

Homotopy trick: de Rham

For

$$\mathrm{d} = \theta^A \frac{\partial}{\partial Z^A}\,.$$

in trivial topology one can set

$$\partial = Z^A \frac{\partial}{\partial \theta^A}\,.$$

This gives

$$A = \theta^A \frac{\partial}{\partial \theta^A} + Z^A \frac{\partial}{\partial Z^A},$$

$$\hat{h} J\,(Z;\theta) = J\,(0;0)\,,$$

$$\mathrm{d}^* J\,(Z;\theta) = Z^A \frac{\partial}{\partial \theta^A} \int_0^1 dt \frac{1}{t} J\,(tZ;t\theta)\,.$$

Z-sector

- Adjoint sector:

$$\Delta^*_{ad} J = -\frac{1}{2i} Z^A \frac{\partial}{\partial \theta^A} \int_0^1 dt \frac{1}{t} \exp\left(-\frac{1-t}{2t} \phi^{BC} \frac{\partial^2}{\partial Y^B \partial \theta^C} \right) J(tZ; Y; t\theta) \,,$$

$$\mathscr{H}_{ad} J(Z, Y, \theta) = \hat{h} \exp\left(-\frac{1}{2} \phi^{AB} \frac{\partial^2}{\partial Y^A \partial \theta^B} \right) J(Z; Y; \theta) \,.$$

- Twisted-adjoint sector

$$\begin{aligned} \Delta^*_{tw} J \quad &:= \quad -\frac{1}{2i} Z^C \frac{\partial}{\partial \theta^C} \int_0^1 dt \frac{1}{t} \exp\left\{ -\frac{i}{8} \left(\frac{1-t}{t} \right)^2 \omega^{AB} h_A{}^C \frac{\partial^2}{\partial \theta^B \partial \theta^C} + i \frac{1-t}{2t} h^{AB} Y_A \frac{\partial}{\partial \theta^B} \right\} \cdot \\ &\quad \cdot \exp\left\{ -\frac{1-t}{2t} \omega^{AB} \frac{\partial^2}{\partial Y^A \partial \theta^B} + \frac{1-t^2}{4t^2} h^{AB} \frac{\partial^2}{\partial Z^A \partial \theta^B} \right\} J(tZ; Y; t\theta) \,. \\ \mathscr{H}_{tw} J \quad &:= \quad \hat{h} \exp\left\{ -\frac{i}{8} \omega^{AB} h_A{}^C \frac{\partial^2}{\partial \theta^B \partial \theta^C} + \frac{i}{2} h^{AB} Y_A \frac{\partial}{\partial \theta^B} \right\} \cdot \\ &\quad \cdot \exp\left\{ -\frac{1}{2} \omega^{AB} \frac{\partial^2}{\partial Y^A \partial \theta^B} + \frac{1}{4} h^{AB} \frac{\partial^2}{\partial Z^A \partial \theta^B} \right\} J(Z; Y; \theta) \,. \end{aligned}$$

Physical sector

- Now one applies resolutions of identity to HS equations in Z-sector

$$\Delta_{ad} f(Z, Y) = J(Z, Y),$$

$$\Delta_{tw} f(Z, Y) = J(Z, Y),$$

resolving all dependence on auxiliary Z, θ-variables.

- After projection onto Z-cohomology one arrives at the equations, describing HS dynamics

$$\mathcal{D}_{ad} f(Y) = J(Y),$$

$$\mathcal{D}_{tw} f(Y) = J(Y).$$

with all functions being space-time forms.

- Can we find a resolution of identity for $\mathcal{D}_{ad}$ and $\mathcal{D}_{tw}$?

Twisted derivative

- Twisted derivative acts on functions of Y as

$$\mathcal{D}_{tw} = \mathcal{D}_L - \frac{i}{2} h^{AB} Y_A Y_B + \frac{i}{2} h^{AB} \frac{\partial^2}{\partial Y^A \partial Y^B},$$

  where $\mathcal{D}_L = \mathrm{d}_X + \omega^{AB} Y_A \frac{\partial}{\partial Y^B}$ - Lorentz-covariant derivative.

- Key observation is that we can take as homotopy operator

$$\partial = -2i Y^A Y^B \nabla_{AB},$$

  where $\nabla_{AB} := \frac{\partial}{\partial h^{AB}}$. It's nilpotent and obeys

$$\left\{ -\frac{i}{2} h^{AB} Y_A Y_B, \partial \right\} = 0.$$

- Also

$$\{\mathcal{D}_L, \partial\} = 0$$

  - if acting on Lorentz tensors. So we request r.h.s. of HS equations to not contain Lorentz connection (see Slava's talk).

Twisted derivative

- Finally one finds

$$A := \{\mathcal{D}_{tw}, \partial\} = h^{\alpha\dot{\beta}}\nabla_{\alpha\dot{\beta}} + y^{\alpha}\bar{y}^{\dot{\beta}}\frac{\partial^2}{\partial y^{\alpha}\bar{y}^{\dot{\beta}}} + h^{\alpha\dot{\beta}}\bar{y}^{\dot{\gamma}}\frac{\partial}{\partial\bar{y}^{\dot{\beta}}}\nabla_{\alpha\dot{\gamma}} + h^{\alpha\dot{\beta}}y^{\gamma}\frac{\partial}{\partial y^{\alpha}}\nabla_{\dot{\beta}\gamma}.$$

  It acts non-diagonally in terms of Y, h-powers due to the last two terms. So one needs more subtle decomposition of the module.

- Proper decomposition is realised through the following representation of functions

$$X = \oplus G\left(h, y, \frac{\partial}{\partial y}\bar{y}, \frac{\partial}{\partial\bar{y}}\right) F\left(y, \bar{y}\right)$$

  where in G all h-dependence is extracted in an irreducible way (i.e. Y and $\partial/\partial Y$ are contracted only with h).

Twisted derivative

- Decomposition

$$X = \oplus G\left(h, y, \frac{\partial}{\partial y}\bar{y}, \frac{\partial}{\partial \bar{y}}\right) F\left(y, \bar{y}\right)$$

  is natural for HS equations, where F contains all dependence on HS fluctuations $C\left(Y\right)$, $\omega\left(Y\right)$

- All possible G can be enumerated. First, we introduce basis forms built of $h^{\alpha\dot{\beta}}$:

$$1,\ h^{\alpha\dot{\beta}},\ H^{\alpha\beta},\ H^{\dot{\alpha}\dot{\beta}},\ H^{\alpha\dot{\beta}},\ H^4.$$

  Then possible G are

$$1,\ h^{\alpha\dot{\beta}}y_\alpha\bar{y}_{\dot{\beta}},\ h^{\alpha\dot{\beta}}y_\alpha\bar{\partial}_{\dot{\beta}},\ h^{\alpha\dot{\beta}}\partial_\alpha\bar{y}_{\dot{\beta}},\ h^{\alpha\dot{\beta}}\partial_\alpha\bar{\partial}_{\dot{\beta}},\ H^{\alpha\beta}y_\alpha y_\beta,\ H^{\alpha\beta}y_\alpha\partial_\beta,\ H^{\alpha\beta}\partial_\alpha\partial_\beta,\ldots$$

  and so on.

Twisted derivative

All relevant information about $X = G\left(h, y, \frac{\partial}{\partial y}, \bar{y}, \frac{\partial}{\partial \bar{y}}\right) F(y, \bar{y})$ is encoded in

- $h^{\alpha\dot{\beta}} \nabla_{\alpha\dot{\beta}} X = r \cdot X$
- $y^{\alpha} \frac{\partial}{\partial y^{\alpha}} X = N \cdot X$
- n – number of y in G
- m – number of $\frac{\partial}{\partial y}$ in G

(and analogously $\bar{N}$, $\bar{n}$, $\bar{m}$).

Twisted derivative

In this terms $A = \{\mathcal{D}_{tw}, \partial\}$ can be written as

$$\begin{aligned} AX &= \frac{1}{4}\left(2N + r - \bar{n} + \bar{m}\right)\left(2\bar{N} + r - n + m\right)X - \\ &\quad - \left(\frac{1}{4}\left(r - n + m\right)\left(r - \bar{n} + \bar{m}\right) + r - n\left(1 + m\right) - \bar{n}\left(1 + \bar{m}\right)\right)X \end{aligned}$$

Direct calculation shows that expression in brackets in 2nd string iz zero for all G except

$$h^{\alpha\dot{\beta}} y_\alpha \bar{y}_{\dot{\beta}}, \qquad H^{\alpha\beta} y_\alpha \partial_\beta \ \text{(and c.c.)}, \qquad H^{\alpha\dot{\beta}} \partial_\alpha \partial_{\dot{\beta}},$$

for which it is 1.

Twisted derivative

Equation $AX = 0$ points on cohomology of twisted representation. They are hiding somewhere in:

- $F(y)\,,\ F(\bar{y})\,,$
- $h^{\alpha\dot{\beta}} y_\alpha \bar{y}_{\dot{\beta}}\,,$
- $h^{\alpha\dot{\beta}} y_\alpha F_{\dot{\beta}}(y)\,,\ h^{\alpha\dot{\beta}} \bar{y}_{\dot{\beta}} F_\alpha(\bar{y})$
- $H^{\alpha\beta} y_\alpha y_\beta F(y)\,,\ \bar{H}^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} F(\bar{y})\,.$

Twisted derivative

For all others X, $AX \neq 0$, we can build A^*, inverting A. Then $\mathcal{D}^*_{tw} = \partial A^*$ inverts twisted derivative

- For most of G, except mentioned four ones,

$$\begin{aligned} \mathcal{D}^*_{tw} X &= -2iY^A Y^B \nabla_{AB} \int_0^1 dt \int_0^1 dp \frac{1}{tp} \cdot \\ &\cdot G\left(\sqrt{tp}h, \frac{t}{\sqrt{p}}y, \frac{\sqrt{p}}{t}\frac{\partial}{\partial y}, \frac{p}{\sqrt{t}}\bar{y}, \frac{\sqrt{t}}{p}\frac{\partial}{\partial \bar{y}}\right) F\left(ty, p\bar{y}\right) \end{aligned}$$

- For $G \in \left\{h^{\alpha\dot{\beta}} y_\alpha \bar{y}_{\dot{\beta}}, \quad H^{\alpha\beta} y_\alpha \partial_\beta \text{ (and c.c.)}, \quad H^{\alpha\dot{\beta}} \partial_\alpha \partial_{\dot{\beta}}\right\}$ there modified Bessel function appears in the measure

$$\begin{aligned} \mathcal{D}^*_{tw} X &= -2iY^A Y^B \nabla_{AB} \int_0^1 dt \int_0^1 dp \frac{1}{tp} I_0\left(2\sqrt{\log t \cdot \log p}\right) \cdot \\ &\cdot G\left(\sqrt{tp}h, \frac{t}{\sqrt{p}}y, \frac{\sqrt{p}}{t}\frac{\partial}{\partial y}, \frac{p}{\sqrt{t}}\bar{y}, \frac{\sqrt{t}}{p}\frac{\partial}{\partial \bar{y}}\right) F\left(ty, p\bar{y}\right) \end{aligned}$$

Adjoint derivative

Adjoint derivative acts as

$$\mathcal{D}_{ad} = \mathcal{D}_L + h^{AB} Y_A \partial_B = \mathcal{D}_L + h^{\alpha\dot\beta} y_\alpha \bar\partial_{\dot\beta} + h^{\alpha\dot\beta} \partial_\alpha \bar y_{\dot\beta} .$$

In this case there is no unique natural homotopy operator. Instead there are two equally good of them, each breaking $sp\,(4)$-symmetry

$$\partial_1 = y^\alpha \bar\partial^{\dot\beta} \nabla_{\alpha\dot\beta} , \qquad \partial_2 = \partial^\alpha \bar y^{\dot\beta} \nabla_{\alpha\dot\beta} .$$

Intersection of $A_1 X = 0$ and $A_2 X = 0$ gives all possible cohomology of adjoint representation

- $F = const$
- $h^{\alpha\dot\beta} F_{\alpha\dot\beta}$
- $H^{\alpha\beta} F_{\alpha\beta}\,(y) , \quad \bar H^{\dot\alpha\dot\beta} F_{\dot\alpha\dot\beta}\,(\bar y)$
- $H^{\alpha\dot\beta} F_{\alpha\dot\beta}$
- H^4

Adjoint derivative

The same analysis as in twisted case leads to inverse operators $\mathcal{D}_1^*$ and $\mathcal{D}_2^*$.
Like in twisted case, there main and Bessel sectors arise

- main sector

$$\mathcal{D}_1^* X = -y^\alpha \bar{\partial}^{\dot\beta} \nabla_{\alpha\dot\beta} \int_0^1 dt \int_0^1 dp \frac{p}{t} G\left(\sqrt{tp}h, \frac{t}{\sqrt{p}}y, \frac{\sqrt{p}}{t}\frac{\partial}{\partial y}, \frac{p}{\sqrt{t}}\bar{y}, \frac{\sqrt{t}}{p}\frac{\partial}{\partial \bar{y}}\right) F\left(ty, p\bar{y}\right)$$

- Bessel sector: $G \in \left\{h^{\alpha\dot\beta} y_\alpha \bar{\partial}_{\dot\beta}, \quad H^{\alpha\beta} y_\alpha \partial_\beta (\text{and c.c.}), \quad H^{\alpha\dot\beta} \partial_\alpha \bar{y}_{\dot\beta}\right\}$

$$\begin{aligned}\mathcal{D}_1^* X \quad &= \quad -y^\alpha \bar{\partial}^{\dot\beta} \nabla_{\alpha\dot\beta} \int_0^1 dt \int_0^1 dp \frac{p}{t} J_0\left(2\sqrt{\log t \cdot \log p}\right) \cdot \\ &\quad \cdot G\left(\sqrt{tp}h, \frac{t}{\sqrt{p}}y, \frac{\sqrt{p}}{t}\frac{\partial}{\partial y}, \frac{p}{\sqrt{t}}\bar{y}, \frac{\sqrt{t}}{p}\frac{\partial}{\partial \bar{y}}\right) F\left(ty, p\bar{y}\right)\end{aligned}$$

Bessel functions

Presented analysis requested inversion of three different type of functions, that was realised using

- $\frac{1}{a \cdot b} = \int_0^1 dt \int_0^1 dp \frac{1}{t \cdot p} t^a p^b$
- $\frac{1}{a \cdot b - 1} = \int_0^1 dt \int_0^1 dp \frac{1}{t \cdot p} I_0 \left(2\sqrt{\log t \cdot \log p} \right) t^a p^b$
- $\frac{1}{a \cdot b + 1} = \int_0^1 dt \int_0^1 dp \frac{1}{t \cdot p} J_0 \left(2\sqrt{\log t \cdot \log p} \right) t^a p^b$

Conclusion

Using homotopy trick, an inverse operators for Lorentz-covariant equations of nonlinear HS theory were constructed. To be done: general formulas unifying main and Bessel sectors, resolutions of identity, formulas for spectral sequences.