Counding Higher-Spin Irreducibles

from String Field Theory

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Each mixed symmetry higher-spin N irreducible representation is described by a certain Young diagram

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At the same time, each k-row diagram can be labeled by an ordered length k partition of number N:

$$\begin{aligned} N = n_1 + n_2 + ... + n_k \\ n_1 \geq n_2 ... \geq n_k > 0 \end{aligned} \tag{1}$$

where $n_1, ...n_k$ are the lengths of the rows. Therefore,the problem of counting all the irreducible representations of spin N is isomorphic to the problem of finding the number of the ordered partitions of N.

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The problems of interest are therefore to count both the total number $\lambda(N)$ of the partitions of N and (more difficult) the number $\lambda(N|k)$ of the restricted partitions of the length k, that determines the number of the mixed symmetry higher-spin irreps with k rows.

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For example, $\lambda(4) = 5, \lambda(5) = 7, \lambda(5|3) = 2$. Both of these problems are well-known in number theory, with no analytic formula for arbitrary N, k available.

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For $\lambda(N)$ some asymptotic expressions are known. The best-known one is by Hardy-Ramanujan (1918):

$$\lambda(N) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$$

with some later improvements of this formula, in particular, by the one proposed by Rademacher (1937) and proved by Erdos (1942) expressing $\lambda(N)$ in terms of the convergent series:

$$\lambda(N) = \frac{1}{\pi\sqrt{2}} \sum_{n=1}^{\infty} \sqrt{n}\alpha_n(N) \frac{d}{dN} \{ \frac{sinh[\frac{\pi}{n}\sqrt{\frac{2}{3}(N-\frac{1}{24})}]}{\sqrt{N-\frac{1}{24}}}$$

where

$$\alpha_n(N) = \sum_{0 \leq m \leq n; [m|n]} e^{i\pi(s(m,n)-\frac{2Nm}{n})}$$

with the notation $(m|n)$ implying the sum over m taken over the values of m relatively prime to n and

$$s(m,n) = \frac{1}{4n} \sum_{k=1}^{n-1} cot(\frac{\pi k}{n}) cot(\frac{\pi km}{n})$$

is the Dedekind sum for co-prime numbers.

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At the same time, no similar relations are known for $\lambda(N|k)$, although it is possible to write down the generating functions both for $\lambda(N)$ and $\lambda(N|k)$:

$$F(x)=\prod_{n=1}^{\infty}\frac{1}{1-x^n}=\sum_{N=0}^{\infty}\lambda(N)x^N$$

$$F(x,y)=\prod_{n=1}^{\infty}\frac{1}{1-yx^n}$$

$$=\sum_{N=0,k=0;k\leq n}^{\infty}\lambda(N|k)x^Ny^k$$

Unfortunately, however, one cannot elucidate the analytic expressions for $\lambda(N)$ and $\lambda(N|k)$ from these generating functions for arbitrary N and k.

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Our purpose is to deduce $\lambda(N)$ and $\lambda(N|k)$ from conformal transformations of certain correlator of irregular vertex operators in string field theory (which also can be understood as generating vertices for higher-spin modes in string theory) such that:

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1. Computed on the upper half-plane, the correlator counts the number of partitions, i.e. reproduces the generating function $F(x,y)$ for $\lambda(N|k)$.

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2. Upon certain suitable conformal transformation (to be identified) the correlator gives a calculable analytic expression, allowing to deduce $\lambda(N|k)$ by using the conformal symmetry. Such is the strategy that we shall follow.

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With some effort, it is straightforward to identify the correlator counting the number $\lambda(N|k)$ of partitions. In the upper half-plane, it is given by

$$G(\alpha,\beta|z,w) =< U_\alpha(z)V_\beta(w) > |_{z=i;w=i\epsilon}$$

where

$$U_\alpha(z) =: \prod_{n=0}^{\infty}\frac{1}{1-\frac{\alpha^n\partial^n\phi}{n!}} : (z)$$
$$V_\beta(w) =: e^{\beta\partial\phi} : (w)$$

where ϕ is $2d$ boson (e.g. a Liouville field or an open string's target space coordinate), $\epsilon \to 0$, α and β are the parameters that are to control the generating function for the partitions. Indeed, expanding in α:

$$U_\alpha = \sum_{k=1}^{\infty}\sum_{n_1\leq\ldots\leq n_k=0}^{\infty}\sum_{p_1,\ldots,p_k}\frac{\alpha^{p_1n_1+p_2n_2+\ldots+p_kn_k}}{(n_1!)^{p_1}\ldots(n_k!)^{p_k}}$$
$$\times : (\partial^{n_1}\phi)^{p_1}\ldots(\partial^{n_k}\phi)^{p_k} :$$

using the OPE:

$$\partial^n\phi(z) : e^{\beta\partial\phi} : (w) \sim \frac{n!\beta : e^{\beta\partial\phi} : (w)}{(z-w)^{n+1}}$$

and introducing $N = \sum p_k n_k$

one easily calculates

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\begin{aligned}
G(\alpha,\beta|\epsilon) &=< U_\alpha(z)V_\beta(w) > |_{z=i;w=i\epsilon} \\
&= \sum_{[N|n_1...n_k]=0}^{\infty} \sum_{k=0}^{N} \frac{\alpha^N \beta^k \lambda(N|k)}{(z-w)^{N+k}} \\
&= \prod_{n=1}^{\infty} \frac{1}{1-\tilde{\alpha^n}\tilde{\beta}} \\
\tilde{\alpha} &= \frac{\alpha}{z-w}; \tilde{\beta} = \frac{\beta}{z-w}
\end{aligned}
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i.e. G is the generating function for restricted partitions with

$$\lambda(N|k) = \frac{i^{N+k}}{N!k!}(1-\epsilon)^{N+k}\partial_\alpha^N \partial_\beta^k G(\alpha,\beta|\epsilon)$$

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Side remark 1:

U_α is an analytic solution of open cubic SFT $QU_\alpha + U_\alpha \star U_\alpha = 0$ for $\alpha \approx 4.6$. Generalizations of U_α to Toda theories ($\phi \to \vec{\phi} = (\phi_1, ..., \phi_D)$) are the solutions for certain values $\alpha = \alpha(D)$, with α related to Λ and the solution interpolating between flat and AdS_D backgrounds (D.P., in progress) U_α also can be regarded as a generating vertex for the higher spin operators in bosonic string.

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Side remark 2:

V_β is a rank 1 irregular vertex operator satisfying

$$L_1 V_\beta = l_1(\beta) V_\beta; L_2 V_\beta = l_2(\beta) V_\beta$$
$$L_n V_\beta = 0 (n > 2)$$

It is physically a "dipole" with β being the dipole's size

- Now that we have identified the correlator generating $\lambda(N|k)$, the next step is to identify the suitable conformal transformation, which turns out to be

$$z \to f(z) = e^{-\frac{i}{z}} \tag{2}$$

(note that this transformation is well-behaved and nonsingular in the upper half-plane)

- Now we have to:

- Compute infinitezimal transformations of U_α and V_β

- Integrate them to get the finite transformations for U_α and V_β under $f(z)$

- Since $f(z)$ is not a fractional-linear transformation, integrate the Ward identities for $f(z)$, to ensure that the correlators match upon $f(z)$.

- Compute the correlator in the new coordinates, in order to obtain the analytic expression for $G(\alpha, \beta|\epsilon)$.

Step 1. Infinitezimal transformations

The straightforward computation using the stress-energy tensor

$$T(z) = \frac{1}{2} : (\partial\phi)^2 :$$

gives:

$$\delta_\epsilon U_\alpha(z) = [\oint \frac{du}{2i\pi}\epsilon(u)T(u); U_\alpha(z)]$$
$$= \sum_{n=1}^{\infty}\{\frac{\alpha^n(\partial^n(\epsilon\partial\phi))}{\prod_{k=1;k\neq n}^{\infty}(1-\frac{\alpha^k\partial^k\phi}{k!})\times(1-\frac{\alpha^n\partial^n\phi}{n!})^2}$$
$$+\frac{2\alpha^{2n}(n!)^2\partial^{2n+1}\epsilon}{(2n+1)!\prod_{k=1;k\neq n}^{\infty}(1-\frac{\alpha^k\partial^k\phi}{k!})\times(1-\frac{\alpha^n\partial^n\phi}{n!})^3}$$
$$+\sum_{0\leq n_1<n_2<\infty}\frac{\alpha^{n_1+n_2}n_1!n_2!\partial^{n_1+n_2+1}\epsilon}{\prod_{k=1;k\neq n_1,n_2}^{\infty}(1-\frac{\alpha^k\partial^k\phi}{k!})\times(1-\frac{\alpha^{n_1}\partial^{n_1}\phi}{n_1!})(1-\frac{\alpha^{n_2}\partial^{n_2}\phi}{n_2!})}$$

and

$$\delta_\epsilon V_\beta = (\beta\partial\epsilon\partial\phi + \frac{1}{12}\beta^2\partial^3\epsilon)V_\beta$$

Integrating these infinitezimal transformations, we obtain the transformations of U_α and V_β for the finite conformal transformation $f(z) = e^{-\frac{i}{z}}$:

$$U_\alpha(z) \to \prod_{n=1}^{\infty}\frac{1}{(1-\frac{\alpha^n\delta_n(\phi,f)}{n!})(1-\frac{\alpha^{2n}S_{n|n}(f;z)}{(1-\frac{\alpha^n\delta_n(\phi,f)}{n!})^2})}$$
$$\times\prod_{0<n_1<n_2<\infty}\frac{1}{\frac{\alpha^{n_1+n_2}S_{n_1|n_2}(f;z)}{(1-\frac{\alpha^{n_1}\delta_{n_1}(\phi,f)}{n_1!})(1-\frac{\alpha^{n_2}\delta_{n_2}(\phi,f)}{n_2!})}}$$

where

$$S_{n_1|n_2}(f;z) = \frac{1}{n_1+n_2+1}B^{(n_1+n_2)}(-log(\frac{df}{dz}))$$
$$-\frac{1}{(n_1+1)(n_2+1)}B^{(n_1)}B^{(n_2)}(-log(\frac{df}{dz}))$$

are the generalized higher-derivative Schwarzians for $log(-f'(z))$,

$$\delta_n(\phi,f) = \frac{\partial^n f}{\partial z^n}\partial\phi(f(z))$$
$$+\sum_{k=1}^{n-1}\sum_{l=1}^{k}\frac{(n-1)!}{(n-1-k)!}\frac{\partial^{n-k}f}{\partial z^{n-k}}B^{k|l}(f)\partial^{l+1}\phi(f(z))$$

$B^{(k)}(f)$ and $B^{k|l}(f)(l{\leq}k)$ are the *complete* and *incomplete* Bell polynomials respectively, defined according to:

$$B^{(k)}(f) = \sum_{l=1}^{k}B^{k|l}(f)$$
$$B^{k|l}(f) = \sum_{k|k_1...k_l}\frac{\partial^{k_1}f...\partial^{k_l}f}{k_1!...k_l!q(k_1)!...q(k_l)!}$$

where $q(k_j)$ are the multiplicities of k_j in the partition $k = k_1+.....+k_l$. Next, the finite transformation of the dipole V_β is:

$$V_\beta(w)|_{w=i\epsilon} \to e^{\beta\frac{\partial f}{\partial z}\phi(f(z))+\frac{\beta^2}{12}S_{1|1}(-log(\frac{\partial f}{\partial z}))}|_{f(z)=e^{-\frac{i}{z}}}$$

where $S_{1|1}$ coincides with the *ordinary* Schwarzian. Explicitly, in the new coordinates

$$V_\beta(i\epsilon) = e^{-\frac{i}{\epsilon^2}e^{-\frac{1}{\epsilon}\partial\phi-\frac{\beta^2}{24\epsilon^4}}}$$

The crucial property of this transformation is that, in the new cooordinates the dipole size shrinks to zero as $\epsilon \to 0$ and in the correlator $< U_\alpha V_\beta >$ all the contractions of derivatives of ϕ with $\partial\phi$ in V_β produce terms that are exponentially suppressed in this limit. Therefore, only the non-contraction contributions, produced by the zero modes of the both operators, survive in this limit. This makes it an easy problem to compute the correlator, despite the U_α-operator by itself looking extremely cumbersome. The final step is to integrate the Ward identities, to ensure that the correlators, related by the conformal transformation $f(z) = e^{-\frac{i}{z}}$ match. In general, integrating the Ward identities is a hard problem, however, for the fiven $f(z)$ the problem simplifies crucially due to the exponential suppressions and the result is reduced to the shift

$$2\alpha^{2n} S_{n|n}(f) \to 2\alpha^{2n} S_{n|n}(f) + \frac{\alpha^n \beta}{n!} S_{1|n}(\tilde{f}) \tag{3}$$

for the generalized Schwarzians, where

$$\tilde{f}(z) = \frac{e^{-\frac{i}{z}}}{z - i\epsilon}$$

The resulting expression for the generating function of the restricted length k partitions of N is

$$G(\alpha, \beta|\epsilon) = \prod_{n;n_1<n_2} \frac{e^{\frac{\beta^2 S_{1|1}(e^{-\frac{i}{z}})}{12}}|_{z=i\epsilon}}{1 - 2\alpha^{2n} S_{n|n}(e^{-\frac{i}{z}}) - \alpha^n \beta S_{1|n}(\frac{e^{-\frac{i}{z}}}{z-i\epsilon})(1 - \alpha^{n_1+n_2} S_{n_1|n_2}(e^{-\frac{i}{z}}))|_{z=i}}$$

Upon differentiating in α, β and regularizing in ϵ

$$\lambda(N|k) = (1+\epsilon)^{N+k} \partial_\alpha^N \partial_\beta^k G(\alpha, \beta, \epsilon)$$

this gives the analytic expression for $\lambda(N|k)$ in terms of finite series in generalized Schwarzians, that can be verified numerically. QED