Various semiclassical limits of torus conformal blocks

Konstantin Alkalaev

Tamm Theory Department, Lebedev Physical Institute

K.Alkalaev, V. Belavin, arXiv:1603.08440
K. Alkalaev, R. Geiko, V. Rappoport, arXiv:1612.05891

Moscow 2017

Outline

- Short review of the sphere case duality
- Semiclassical conformal blocks on a torus
- Symmetry argument: Virasoro algebra contractions
- Holographic interpretation of the classical torus block
- Conclusions and outlooks

Heavy and light operators

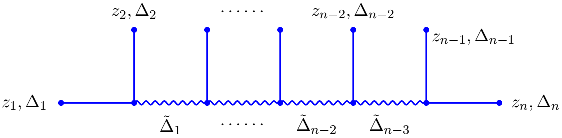
The n-point correlation function of $V_{\Delta_i}(z_i)$, $i = 1, ..., n$ are given by

$$\langle V_{\Delta_1}(z_1) \dots V_{\Delta_n}(z_n) \rangle \sim \sum_{\tilde{\Delta}} C ... C \, \mathcal{F}$$

Conformal blocks

$$\mathcal{F}(z_1, ..., z_n | \Delta_1, ..., \Delta_n; \tilde{\Delta}_1, ..., \tilde{\Delta}_{n-3} | c)$$

are conveniently depicted as (in a particular OPE channel)



Different semiclassical limits of the conformal blocks depend on the behavior of Δ_i and $\tilde{\Delta}_i$.

- $\Delta, \tilde{\Delta} = \mathcal{O}(c^1)$: heavy operators
- $\Delta, \tilde{\Delta} = \mathcal{O}(c^0)$: light operators

Three types of blocks:

- Global conformal block — all operators are light
- Classical conformal block — all operators are heavy
- Heavy-light blocks interpolate between these two extreme regimes

Classical conformal block

In the semiclassical limit $c \to \infty$ the conformal blocks exponentiate as (Zamolodchikov 1986)

$$\mathcal{F}(z_i|\Delta_i, \tilde{\Delta}_j) = \exp\big[- c\, f(z_i|\epsilon_i, \tilde{\epsilon}_j)\big]$$

where $\epsilon_k = \frac{\Delta_k}{c}$ and $\tilde{\epsilon}_k = \frac{\tilde{\Delta}_k}{c}$ are *classical dimensions* and $f(z|\epsilon, \tilde{\epsilon})$ is the *classical* conformal block.

Heavy-light perturbation expansion (Fitzpatrick, Kaplan, Walters' 2014) We consider the case of *two* background operators. Let $\epsilon_{n-1} = \epsilon_n \equiv \epsilon_h$ be the *background heavy* dimension, while ϵ_i, $i = 1, \ldots, n-2$ be *perturbative heavy* dimensions,

$$\frac{\epsilon_i}{\epsilon_h} \ll 1$$

The classical conformal block is expanded as

$$f(z|\epsilon, \tilde{\epsilon}) = f^{(0)}(z|\epsilon, \tilde{\epsilon}) + f^{(1)}(z|\epsilon, \tilde{\epsilon}) + f^{(2)}(z|\epsilon, \tilde{\epsilon}) + \ldots\,,$$

The zeroth approximation corresponds to the classical conformal block of the 2-point function of the background operators. Hence, $f^{(0)} = 0$ and the first non-trivial correction is given by $f^{(1)}$.

Geodesic networks

The background heavy operators $\epsilon_n = \epsilon_{n-1} \equiv \epsilon_h$ produce an asymptotically AdS_3 geometry identified either with an angular deficit or BTZ black hole (Fitzpatrick, Kaplan, Walters 2014)

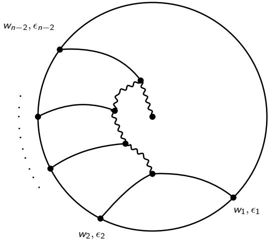
$$\alpha = \sqrt{1 - 4\epsilon_h}$$

Here

- $\alpha^2 > 0$ for an angular deficit
- $\alpha^2 < 0$ for the BTZ black hole

In the case of the angular deficit the metric reads

$$ds^2 = \alpha^2 \sec^2 \rho \,(dt^2 + \sin^2 \rho d\phi^2 + \alpha^{-2} d\rho^2)$$



The perturbative heavy operators are realized via particular network of worldlines of point-like probes propagating in the background geometry formed by the two background heavy operators. Points w_i are boundary attachments of the perturbative heavy operators.

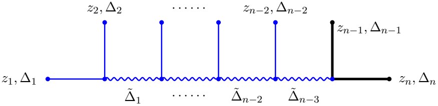
See, e.g., (Hartman 2013, Asplund, Bernamonti, Galli, Hartman 2014, Caputa, Simon, Stikonas, Takayanagi 2014, Hijano, Kraus, Snively 2015, Alkalaev, Belavin 2015, Banerjee, Datta, Sinha 2016, Chen, Wu, Zhang 2016)

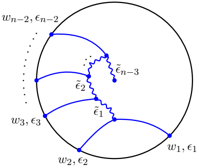
The block/length correspondence

$$f^{(1)} \cong S_{bulk} \,, \qquad \text{where} \qquad S_{bulk} = \sum_{i=1}^{n-2} \epsilon_i \, L_i + \sum_{i=1}^{n-3} \tilde{\epsilon}_i \, \tilde{L}_i \,,$$

and L_i and $\tilde{L}_i$ are lengths of different geodesic segments on a fixed time slice.

Large-c duality: sphere case





$$f^{(1)}(z|\epsilon, \tilde{\epsilon}) \;\; \cong \;\; S(w|\epsilon, \tilde{\epsilon})$$

Going beyond the spherical CFT: 1-point torus block

- Extend to Riemannian surfaces of genus g!
- Non-trivial 0-point and 1-point functions already on a torus
- Torus 1-point block $\sim$ sphere 4-point block (Fateev, Litvinov, Neveu, Onofri 2009, Poghossian 2009)
- Classical conformal blocks with any n and g? Quantum conformal blocks with any n and g (Cho, Collier, Yin 2017)

Virasoro 1-point torus block

The 1-point torus correlation function is

$$\langle \phi_\Delta(z,\bar{z}) \rangle \sim \sum_{\tilde{\Delta}} C_{\Delta\tilde{\Delta}\tilde{\Delta}} \mathcal{V}(z|q)\mathcal{V}(\bar{z}|\bar{q})$$

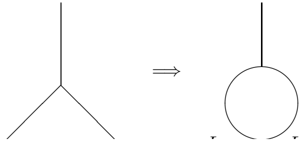
(Holomorphic) 1-point conformal block is

$$\mathcal{V}(\Delta,\tilde{\Delta},c|q) = q^{\tilde{\Delta}-c/24} \sum_{n=0}^{\infty} q^n \mathcal{V}_n(\Delta,\tilde{\Delta},c)\ , \qquad \text{where} \qquad q = e^{2\pi i \tau}\ ,$$

is the elliptic parameter on a torus with the modulus τ, and the expansion coefficients are

$$\mathcal{V}_n(\Delta,\tilde{\Delta},c) = \frac{1}{\langle \tilde{\Delta}|\phi_\Delta(z)|\tilde{\Delta}\rangle} \sum_{n=|M|=|N|} B^{M|N} \langle \tilde{\Delta}, M|\phi_\Delta(z)|N,\tilde{\Delta}\rangle\ ,$$

where $|\tilde{\Delta}, M\rangle = L^{i_1}_{-m_1} \ldots . L^{i_k}_{-m_k}|\tilde{\Delta}\rangle$ are descendant vectors in the Verma module generated from the primary state $|\tilde{\Delta}\rangle$. Here, M labels basis monomials, $|M| = i_1 m_1 + \ldots + i_k m_k$ denotes the sum of the Virasoro generator indices. The matrix $B^{M|N}$ is the inverse of the Gram matrix $B_{M|N} = \langle \tilde{\Delta}, M|N, \tilde{\Delta}\rangle$. Tadpole diagram:



Semiclassical torus blocks

Large-c expansions of the block function $\mathcal{V}(\Delta, \tilde{\Delta}, c|q)$. *Four* types of blocks:

- $\Delta, \tilde{\Delta} = \mathcal{O}(c^0)$. In the limit $c \to \infty$ we distinguish between the global and light blocks. The light block is the leading asymptotic in the c-recursive representation of the torus block.
- $\Delta = \mathcal{O}(c^0)$ and $\tilde{\Delta} = \mathcal{O}(c^1)$. In the limit $c \to \infty$ we obtain the heavy-light block. It follows that

  $$\Delta/\tilde{\Delta} \ll 1$$

  The opposite regime of heavy external operator and light exchanged operator does not exist! The vacuum approximation (Hartman 2013) does not exist as well.
- $\Delta, \tilde{\Delta} = \mathcal{O}(c^1)$. The large-$c$ expansion is the exponential of the classical conformal block.
- $\Delta, \tilde{\Delta} = \mathcal{O}(c^1)$ and $\Delta/\tilde{\Delta} \ll 1$. This is the linearized classical conformal block (we will describe the holographic interpretation in the thermal AdS_3 space).

All four types of semiclassical torus blocks are connected by various links!

Global torus block

- A finite-dimensional subalgebra $sl(2,\mathbb{C}) \subset Vir$. The associated block is *global*.
- Does not depend on the central charge. Note that the general conformal block of higher dimensional CFT being restricted to two dimensions yields the global block.

The general Virasoro block simplifies to yield

$$\mathcal{F}(\Delta,\tilde{\Delta}|q) = \sum_{n=0}^{\infty} q^n \mathcal{F}_n(\Delta,\tilde{\Delta})\;, \qquad \mathcal{F}_n(\Delta,\tilde{\Delta}) = \frac{1}{\langle\tilde{\Delta}|\phi_\Delta(z)|\tilde{\Delta}\rangle}\frac{\langle\tilde{\Delta}|L_1^n\phi_\Delta(z)L_{-1}^n|\tilde{\Delta}\rangle}{\langle\tilde{\Delta}|L_1^nL_{-1}^n|\tilde{\Delta}\rangle}\;,$$

where $L_{0,\pm1}$ are $sl(2,\mathbb{C})$ basis elements. The coefficients can be packed into the hypergeometric function

$$\mathcal{F}(\Delta,\tilde{\Delta}|q) = (1-q)^{-\Delta}\;{}_2F_1(2\tilde{\Delta}-\Delta, 1-\Delta, 2\tilde{\Delta}\,|\,q)$$

In the the limiting case $\Delta \to 0$ we arrive at the zero-point block coinciding with the $sl(2,\mathbb{C})$ character,

$$\mathcal{F}(\Delta,\tilde{\Delta}|q)\Big|_{\Delta\to 0} = \frac{1}{1-q} = 1+q+q^2+q^3+\ldots\;,$$

which indicates that there is just one state on each level of the corresponding $sl(2,\mathbb{C})$ module.

Large dimension expansion

The second order ODE for the global block

$$\mathcal{F}'' + 2\left(\frac{\tilde{\Delta}}{q} - \frac{1}{1-q}\right)\mathcal{F}' - \left(\frac{\Delta(\Delta-1)}{q(1-q)^2} + \frac{2\tilde{\Delta}}{q(1-q)}\right)\mathcal{F} = 0$$

We consider the regime where dimensions Δ and $\tilde{\Delta}$ tend to infinity in a coherent manner:

$$\Delta = \kappa\,\sigma \qquad \tilde{\Delta} = \kappa\,\tilde{\sigma} \qquad \text{where} \quad \sigma, \tilde{\sigma} \quad \text{are rescaled conformal dimensions}$$

Proposition. *The large-k asymptotic expansion of the global block function is exponentiated*

$$\mathcal{F}(\Delta, \tilde{\Delta}|q) = \exp\left[\kappa\, g(\sigma, \tilde{\sigma}|q)\right]\,, \qquad \kappa \to \infty$$

where $g(\sigma, \tilde{\sigma}|q)$ is formally given by

$$g(\sigma, \tilde{\sigma}|q) = \int_0^q dx\left(-\frac{\tilde{\sigma}}{x} + \sqrt{\frac{\tilde{\sigma}^2}{x^2} + \frac{\sigma^2}{x(1-x)^2}}\right)$$

Comments:

- Global sphere blocks satisfy the $sl(2)$ Casimir equation (Dolan, Osborn 2003).
- Expansion in $\sigma/\tilde{\sigma}$ yields the linearized classical torus block.

Light torus block

In general, the light block can be defined as the $c \to \infty$ limit of the quantum conformal block. We expand conformal blocks given that

$$\Delta = \mathcal{O}(c^0) \qquad \text{and} \qquad \tilde{\Delta} = \mathcal{O}(c^0) \qquad \text{as} \qquad c \to \infty$$

In this regime the quantum torus block is

$$q^{c/24-\tilde{\Delta}}\, \mathcal{V}(\Delta, \tilde{\Delta}, c|q) = \mathcal{L}(\Delta, \tilde{\Delta}|q) + \mathcal{O}(c^{-1})$$

where the leading term is the one-point light torus block.

Proposition. *The global and light blocks are related as*

$$\mathcal{L}(\Delta, \tilde{\Delta}|q) = \frac{1-q}{\varphi(q)}\, \mathcal{F}(\Delta, \tilde{\Delta}|q)\ ,$$

where $\varphi(q) = \prod_{n=1}^{\infty}(1-q^n)$ *is the Euler function.*

- The c-recursive relations for 1-pt torus blocks (Alkalaev, Rappoport, Geiko 2016)
- The factorization property persists for any n and g (Cho, Collier, Yin 2017)

Virasoro algebra contractions

Virasoro algebra commutation relations are given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\ , \qquad m,n \in \mathbb{Z}$$

while primary operators transform as

$$[L_m, \phi_\Delta] = z^m(z\partial_z + (m+1)\Delta)\phi_\Delta$$

Inönu-Wigner contraction for the Virasoro algebra, where the deformation parameter is c^{-1}.

Rescaled Virasoro generators are $L_m \to c^{-\gamma(m)}L_m$, where $\gamma(m)$ is some function of $m \in \mathbb{Z}$. There are two choices:

- Case (A): $\quad L_{0,\pm1} \to l_{0,\pm1} = L_{0,\pm1}\ , \qquad L_m \to a_m = L_m/c^1\ , \quad |m| \geq 2$
- Case (B): $\quad L_{0,\pm1} \to l_{0,\pm1} = L_{0,\pm1}\ , \qquad L_m \to a_m = L_m/c^{1/2}\ , \quad |m| \geq 2$

The transformation law is also rescaled. In the limit $c \to \infty$, keeping the conformal dimension Δ finite we find that in both cases (A) and (B) the primary operator transforms as

$$[l_m, \phi_\Delta] = z^m(z\partial_z + (m+1)\Delta)\phi_\Delta\ , \qquad [a_m, \phi_\Delta] = 0\ .$$

It follows that ϕ_Δ are $sl(2)$ conformal operators and a_m-singlets.

- Case (A): the contracted Virasoro algebra splits into $sl(2)$ algebra and the infinite-dimensional Abelian algebra $\mathcal{A}$,

$$[l_m, l_n] = (m-n) l_{m+n} \;, \qquad [a_m, a_n] = 0 \;,$$

$$[l_m, a_n] = (m-n) a_{m+n} \;, \qquad |m+n| \geq 2 \;; \qquad [l_m, a_n] = 0 \;, \qquad |m+n| \leq 1 \;.$$

It is the semidirect sum $Vir_A = sl(2) \ltimes \mathcal{A}$, while the $\pm$ branches of $\mathcal{A}$ are lowest weight $sl(2)$–modules.

- Case (B): the contracted Virasoro algebra splits into $sl(2)$ algebra and the inf-dimensional Heisenberg algebra $\mathcal{H}$,

$$[l_m, l_n] = (m-n) l_{m+n} \;, \qquad [a_m, a_n] = \frac{m(m^2-1)}{12} \delta_{m+n,0} \;,$$

$$[l_m, a_n] = (m-n) a_{m+n} \;, \quad |m+n| \geq 2 \;; \qquad [l_m, a_n] = 0 \;, \quad |m+n| \leq 1 \;.$$

It is the semidirect sum $Vir_B = sl(2) \ltimes \mathcal{H}$, while the $\pm$ branches of $\mathcal{H}$ are lowest weight $sl(2)$–modules.

Proposition. *The contracted algebras Vir_A and Vir_B underlie the global and light torus blocks:*

Case A: Global torus block

Case B: Light torus block

Comments:

- We note that truncating the Virasoro algebra to the $sl(2)$ subalgebra is equivalent to considering the contracted Vir_A with trivially realized Abelian factor.
- The Heisenberg factor $\mathcal{H}$ is non-trivially realized and the global block associated to the $sl(2)$ factor and the block of the full $Vir_B = sl(2) \ltimes \mathcal{H}$ are related.

Heavy-light torus block

The heavy-light limit is defined by

$$\Delta = \mathcal{O}(c^0) \qquad \text{and} \qquad \tilde{\Delta} = \mathcal{O}(c^1) \qquad \text{as} \qquad c \to \infty$$

The heavy-light torus block is defined as

$$\mathcal{H}(\Delta, \tilde{\epsilon}\,|q) = \lim_{\substack{c\to\infty \\ \Delta,\tilde{\epsilon}-\text{fixed}}} q^{c/24-\tilde{\Delta}}\,\mathcal{V}(\Delta, c\tilde{\epsilon}, c\,|q) = 1 + q + 2q^2 + 3q^3 + 5q^4 + \ldots = \varphi^{-1}(q)$$

Proposition. *The heavy-light block is the limiting case of the light block*

$$\mathcal{H}(q) = \lim_{\tilde{\Delta}\to\infty} \mathcal{L}(\Delta, \tilde{\Delta}|q)\,.$$

Comments:

- The heavy-light block is just the zero-point block, or, equivalently, the Virasoro character.
- Similar to the 4-pt sphere blocks (Fitzpatrick, Kaplan, Walters 2015).

The heavy-light block is related to the particular contracted Virasoro algebra

$$L_0 \to l_0 = L_0/c\,, \qquad L_m \to l_m = L_m/\sqrt{2c}\,, \quad m \neq 0\,,$$

In the limit $c \to \infty$ the contracted Virasoro algebra commutation relations read

$$[l_m, l_n] = m\delta_{m+n,0}\, l_0 + \frac{m(m^2-1)}{24}\delta_{m+n,0}\,, \qquad [l_m, \phi_\Delta(z)] = 0\,, \qquad m, n \in \mathbb{Z}$$

The contracted Virasoro algebra is the infinite-dimensional Heisenberg algebra $\mathcal{H}$.

Classical torus block

Now all conformal dimensions grow linearly with the central charge

$$\Delta = \mathcal{O}(c^1) \qquad \text{and} \qquad \tilde{\Delta} = \mathcal{O}(c^1)$$

The Laurent series around $c = \infty$ reads

$$\mathcal{V}(\Delta, \tilde{\Delta}, c|q) = \sum_{n \in \mathbb{N}} \frac{v_n(\epsilon, \tilde{\epsilon}|q)}{c^n} \qquad \text{where finite parameters} \qquad \epsilon = \frac{\Delta}{c} \quad \text{and} \quad \tilde{\epsilon} = \frac{\tilde{\Delta}}{c}$$

are classical conformal dimensions, and $v_n(\epsilon, \tilde{\epsilon}|q)$ are formal power series in the modular parameter q with expansion coefficients being rational functions in ϵ and $\tilde{\epsilon}$.

Exponentiation hypothesis. At large c the principle part goes to zero. Less obvious is the fact that the regular part exponentiates. It follows that the one-point torus block is asymptotically equivalent to

$$\mathcal{V}(\Delta, \tilde{\Delta}, c|q) \sim \exp\Big[\, c\, f(\epsilon, \tilde{\epsilon}|q)\Big] \qquad \text{at} \qquad c \gg 1$$

Function $f(\epsilon, \tilde{\epsilon}|q)$ is the classical conformal block

$$f(\epsilon, \tilde{\epsilon}|q) = (\tilde{\epsilon} - 1/4)\log q + \sum_{n=1}^{\infty} q^n \mathrm{f}_n(\epsilon, \tilde{\epsilon}) = (\tilde{\epsilon} - 1/4)\log q + \frac{\epsilon^2}{2\tilde{\epsilon}}\, q + \ldots$$

Comments:

- Global, light, and heavy-light blocks ($c \to \infty$), classical block ($c \gg 1$).
- Contracted Virasoro algebra + deformations in small c^{-1}

Linearized classical torus block

The torus one-point linearized classical block is defined by introducing the lightness parameter $\delta = \Delta/\tilde{\Delta} \ll 1$. Change from $(\epsilon, \tilde{\epsilon})$ to $(\delta, \tilde{\epsilon})$. We find a double series expansion in δ and $\tilde{\epsilon}$ and fix the terms at most linear in $\tilde{\epsilon}$: the linearized block

$$f(\epsilon, \tilde{\epsilon}|q) = f^{lin}(\delta, \tilde{\epsilon}|q) + \mathcal{O}(\tilde{\epsilon}^2)$$

where

$$f^{lin}(\delta, \tilde{\epsilon}|q) \equiv (\tilde{\epsilon} - 1/4)\log q + \tilde{\epsilon}\sum_{n=1}^{\infty} f_n^{(1)}(q)\delta^{2n}$$

Conjecture. *Provided that $(\kappa, \sigma, \tilde{\sigma})$ of the global block changes to $(c, \epsilon, \tilde{\epsilon})$ of the classical block, the linearized block is related to the exponential factor of the global block at large dimensions*

$$g_0(\epsilon, \tilde{\epsilon}|q) = f^{lin}(\epsilon, \tilde{\epsilon}|q) - (\tilde{\epsilon} - 1/4)\log q\ .$$

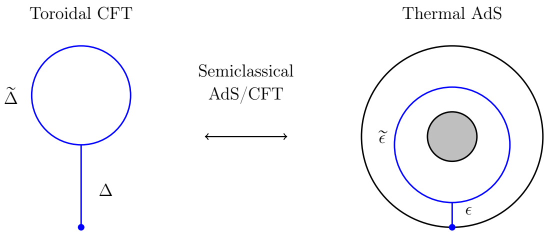
Comments:

- Conjectured integral form for the linearized classical block

$$f^{lin}(\epsilon, \tilde{\epsilon}|q) = \int_0^q dx\left(-\frac{\tilde{\epsilon}}{x} + \sqrt{\frac{\tilde{\epsilon}^2}{x^2} + \frac{\epsilon^2}{(1-x)^2 x}}\right)$$

- The proposition is analogous to the sphere case (Fitzpatrick, Kaplan, Walters 2015, Alkalaev, Belavin 2015)

Large-c duality: torus case



$$-f^{lin} = S_{thermal} + \tilde{\epsilon}\, S_{loop} + \epsilon\, S_{leg}$$

Concluding comments

Conclusions

- There are four different types of semiclassical torus blocks related to each other by a chain of connections. The Virasoro algebra contractions underlie three of them. (The same should be in the sphere case.)

- Classical torus blocks can be expanded around the heavy exchanged channel. The holographic dual is the tadpole graph stretched on the thermal AdS space.

Outlooks

- Higher-point torus blocks in the semiclassical regime
- The monodromy method for torus blocks and its holographic counterpart
- Understand classical blocks using the symmetry argument
- The semiclassical correspondence considered along these lines can be extended by $1/c$ corrections. The 4-point sphere case was studied in (Beccaria, Fachechi, Macorini 2015, Fitzpatrick, Kaplan 2016)