Fefferman-Graham construction and higher-spin fields

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Based on:

X. Bekaert, M.G., E. Skvortsov – work in progress

MG 2012, 2006

K. Alkalaev, M.G., E. Skvortsov 2014

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Warm-up: Ambient space

$\mathbb{R}^{d+2}$ where $o(d,2)$ acts by infinitesimal isometries

Dirac, Thomas, Cartan,...

X^A ($A = +, -, 0, 1, 2, \ldots, d-1$) coordinates on $\mathbb{R}^{d,2}$ where

$\eta_{+-} = 1 = \eta_{-+}, \qquad \eta_{ab} = \text{diag}(-1, +1, \ldots, +1) \qquad a, b = 0, 1, 2, \ldots, d-$

Useful notations $X \cdot Y = \eta_{AB} X^A Y^B$ and $X^2 = X \cdot X$.

AdS space $X^2 = -1$. Explicit embedding

$$X^+ = \rho^{-\frac{1}{2}}, \quad X^- = -\frac{1}{2}(\rho + x^a x_a)\,\rho^{-\frac{1}{2}}, \quad X^a = \rho^{-\frac{1}{2}}\, x^a$$

Ambient representation of $(\nabla^2 - m^2)\varphi = 0$:

$$\left(X \cdot \frac{\partial}{\partial X} + \Delta\right)\Phi = 0\,, \qquad \Box\Phi = 0$$

$$m^2 = \Delta(\Delta - d)$$

For Φ defined for $X^2 < 0$ this is equivalent to $(\nabla^2 - m^2)\varphi = 0$ with $\varphi = \Phi|_{X^2=-1}$.

Frondsal fields on AdS:

$$(\nabla^2 - \ldots)\phi_{\mu_1\ldots\mu_s} = 0\,, \qquad \nabla^\mu \phi_{\mu\mu_2\ldots\mu_s = 0}$$

$$\phi^\mu_{\mu\nu_3\ldots\mu_s} = 0\,, \qquad \delta_\epsilon \phi_{\mu_1\ldots\mu_s} = \nabla_{(\mu_1}\epsilon_{\mu_s\ldots\mu_s)}$$

Ambient picture: $\Phi = \frac{1}{s!}\Phi^{A_1\ldots A_s}(X) P_{A_1}\ldots P_{A_s}$

$$(\partial_P \cdot \partial_P)\Phi = (\partial_P \cdot \partial_X)\Phi = \Box\Phi = 0$$

$$(X \cdot \partial_X - P \cdot \partial_P + 2)\Phi = (X \cdot \partial_P)\Phi = 0\,, \qquad \Phi \sim \Phi + P \cdot \partial_X \epsilon,\,,$$

On-shell conformal scalar

Conformal space: space of rays in the hypecone $X^2 = 0$ in the ambient space. It carries flat conformal structure. Can be identified with conformal compactification of Minskowski space.

The following system:

$$\partial_X \cdot \partial_X \Phi = 0\,, \qquad (X \cdot \partial_X + \frac{d}{2} - l)\Phi = 0\,, \qquad \Phi \sim \Phi + (X^2)^l \lambda$$

in the ambient space $\mathbb{R}^{d+2}$ is equivalent to

$$(\Box_0)^l \phi = 0$$

on d-dimensional Minskowski. The case of $l > 1$ as well as CHS fields was in

Bekaert, MG (2013).

Fefferman-Graham ambient metric

Fefferman-Graham, 1985

On a $d+2$-dimensional manifold $\widehat{\mathcal{X}}$ (think of it as a sort of bundle over $d+1$-dimensional space or d-dimensional conformal space) consider a metric and a vector field:

$$G_{AB}(X)\,, \qquad V^B(X)$$

satisfying

$$L_V G_{AB} = 2G_{AB}\,, \qquad \nabla_A V_B = G_{AB}$$

(∇ – Levi-Civita connection) This data induces a generic metric on the “curved hyperboloid” $V^A V_A = -1$ and vice versa.

In the field-theoretic language:
$L_V G = 2G$, $\nabla_A V_B = G_{AB}$ and ambient diffeomorphisms is equivalent to off-shell gravity on the “curved hyperboloid” $V^A V_A = -1$.

The complete system:

$$L_V G = 2G\,, \qquad \nabla_A V_B = G_{AB}\,, \qquad Ric(G) = 0$$

This data induces: *Fefferman-Graham, 1985*

1) Einstein metric on the curved hyperboloid (cf. $\partial_X \cdot \partial_X \Phi = 0, L_V \phi = -\Delta\phi$ induces $(\nabla^2 - \Delta(\Delta - d))\phi = 0$

2) Off-shell conformal gravity (CGR) for $d = 2k+1$ or on-shell CGR for $d = 2k$ (the CGR equations of motion arise as holograpic Weyl anomaly) on the quotient of the "curved hypercone" $V^2 = 0$.

Original motivation – construction of conformal invariants from ambient Riemannian invariants

Combining 1) and 2) is an ambient tool for near-boundary analysis of the Einstein equations on AdS_{d+1}.

In the case of free HS fields:

techinically improved version of 1)+2) gives a powerful method to study bundary values of AdS gauge fields. In particular, it gives a procedure to exract explicit conformal equations from a simple ambient form of the AdS ones:

Bekaert, MG (2012,2013), Chekmenev, MG (2016)

Alternative (dual) way to describe off-shell CGR:

$$L_V G = 2G\,, \qquad \nabla_A V_B = G_{AB}\,,$$
$$G_{AB} \sim G_{AB} + V^2 \lambda_{AB}\,, \qquad G_{AB} \sim G_{AB} + V_{(A}\lambda_{B)}\,,$$
$$G_{AB} \sim G_{AB} + G_{AB}\lambda\,,$$

where $\lambda_{AB}, \lambda_A, \lambda$ satisfy certain compatibility conditions. Above can be thought of as gauge transformations.

In the context of conformal geometry interpretation (analysys in the vicinity of $V^2 = 0$)

$Ric(G) = 0$ – gauge-fixing condition for the above symmetries

Toy model

Dual Ambient descriptions of off-shell conformal scalar ($\Delta \neq \frac{d}{2} - l$)

$$\partial_X \cdot \partial_X \Phi = 0\,, \qquad (X \cdot \partial_X + \Delta)\Phi = 0$$

$$(X \cdot \partial_X + \Delta)\Phi = 0\,, \qquad \Phi \sim \Phi + X^2\lambda$$

$d_X \cdot \partial_X \Phi = 0$ is a gauge-fixing condition for the gauge equivalence $\Phi \sim \Phi + X^2\lambda$

Straitforward extension to linearized spin-2 and HS fields.

At the linear level: $\Box\Phi = 0$ is an analog of $Ric(G) = 0$. Nonlinear HS version?

sp(2)-form

A simple observaton (though not so well-known:)

Bars, Bonezzi+Latini+Waldron ,...

Working in terms of the invers ambient metric $G^{AB}(X)$ introduce an extra ambient field $\widetilde{G}(X)$ in addition to G^{AB}, V^A.

Using the auxiliary variables P_A (momenta conjugate to X^A so that $\left\{X^A, P_B\right\} = \delta^A_B$) introduce generating functions:

$$F_1 = \frac{1}{2}G^{AB}P_AP_B\,, \quad F_2 = V^AP_A\,, \quad F_3 = \widetilde{G}$$

It turns out that FG relations $L_GV = 2G, \nabla_A V_B = G_{AB}$ are equivalent to

$$\{F_2, F_1\} = 2F_1\,, \qquad \{F_2, F_3\} = -2F_3\,, \qquad \{F_1, F_3\} = F_2$$

in particular it follows $F_3 = -\frac{1}{2}V^2$ and is an auxiliary field.

The system has a natural interpretation in terms of constrained Hamiltonian systems if one interprets F_i as first class constraints of a system whose configuration space is an ambient space:

$$F_i(X,P) = 0, \qquad \left\{F_i, F_j\right\} = C_{ij}^k F_k \,.$$

An infinitesimal canonical transformation

$$F_i \sim F_i + \{F_i, \epsilon\} \;, \qquad \epsilon = \epsilon^A(X) P_A$$

is a natural gauge equivalence (these are ambient diffeomorphisms).

Extra natural symmeries:

$$F_i \sim F_i + \lambda_i^j F_j$$

corresponds to an infinitesimal redefinition of the constraints (which preserve the constraint surface). In this case it is more natural to require just that F_i ar efirst class.

In a similar context field theories associated to constrained systems were put forward in *MG (2006)*

Identifying $sp(2)$-version of FG relation supplemented with gauge symmetries $G^{AB} \sim G^{AB} + \lambda V^2, \ldots$ with the relations and symmetries of a constrained system gives a simple “physical” proof that this system indeed describes off-shell CGR. Indeed, the ambient constraint system

$$F_1^0 = \frac{1}{2}P^2, \quad F_2^0 = X \cdot P, \quad F_3^0 = -\frac{1}{2}X^2$$

is equivalent to

$$H^0(x,p) = p^2$$

in d-dimensions. The deformations of the later one are described by trivial equations and the following gauge transformations of $H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu$

$$\delta H = \{H, \epsilon\} + H\lambda \qquad \delta g_{\mu\nu} = L_\xi g^{\mu\nu} + \omega g^{\mu\nu}$$

This is a spin-2 and Poisson bracket version of the system:

Segal 2002, related earlier work: *Tseytlin, 2002*)

HS extension:

$$F_1 = \Phi + \Phi^A P_A + \frac{1}{2} G^{AB} P_A P_B + \Phi^{ABC} P_A P_B P_C + \dots ,$$
$$F_2 = V^A P_A + \dots ,$$
$$F_3 = \widetilde{G} + \dots$$

$$\{\,\cdot\,,\,\cdot\,\} \quad \to \quad [\cdot,\cdot]_\star$$

and impose the $sp(2)$-form of FG relations

$$[F_i, F_j]_\star = C_{ij}^k F_k \,, \qquad \delta F_i = [F_i, \epsilon]_\star$$

To see that we are on a right track:

Linearize around the vacuum solution:

$$F_1^0 = \frac{1}{2}P^2, \qquad F_2^0 = X \cdot P, \qquad F_3^0 = -\frac{1}{2}X^2$$

$F_i = F_i^0 + f_i$. The linearized gauge symmetries for $f_{2,3}$

$$\delta f_2 = [F_2^0, \epsilon]_\star = (P \cdot \partial_P - X \cdot \partial_X)\epsilon\,, \qquad \delta f_3 = [F_3^0, \epsilon]_\star = X \cdot \partial_P \epsilon\,,$$

In the vicinity of the hyperboloid this implies that the gauge $f_2 = f_3 = 0$ is reachable.

In this gauge linearized $sp(2)$-relations

$$[F_i^0, f_j]_\star + [f_i, F_j^0]_\star = C_{ij}^k f_k$$

imply:

$$X \cdot \partial_P f_1 = 0\,, \qquad (P \cdot \partial_P - X \cdot \partial_X - 2) f_1 = 0$$

while the redisual gauge symmetries are

$$\delta f_2 = [F_2^0, \epsilon]_\star = -P \cdot \partial_X \epsilon \qquad X \cdot \partial_P \epsilon = 0\,, \quad (P \cdot \partial_P - X \cdot \partial_X)\epsilon = 0$$

Global reducibilities: $P \cdot \partial_X \epsilon_0 = X \cdot \partial_P \epsilon_0 = 0$. i.e. we get off-shell HS algebra.

Parent formulation:

In the ambient space introduce Y^A variables (seen as coordinates on the tangent spaces). Then

$$[F_i, F_j]_\star = C_{ij}^k F_k$$

is equivalent to *MG (2012,2006)*

$$dA + \frac{1}{2}[A, A]_Y = 0\,, \qquad dA + [A, F_i]_Y = 0\,, \qquad [F_i, F_j]_Y = C_{ij}^k F_k$$

where $F_i = F_i(X, P, Y)$, $A = dX^B A_B(X, P, Y)$. This can be arrived at by applying Fedosov quantization to the original constrained system. $u(1)$-verison was already in *Vasileiv (2005)*

Advantage: can be considered either in $d+1$-dimensions or in d-dimensions. To recover AdS_{d+1} or conformal system the vacuum should be taken as

$$F_1 = \frac{1}{2}P^2\,, \qquad F_2 = (Y^A + V_0^A)P_A \qquad F_3 = -\frac{1}{2}(Y + V_0)^2\,,$$

where $V_0^A = const$ is a compensator satisfying $V_0^2 = 0$ (for conformal) or $V_0^2 = -1$ for AdS.

To describe off-shell CHS one has to employ extra symmetries (re-definition of the constraints):

$$A \sim A + \chi^i * F_i \,, \qquad F_i \sim F_i + \lambda_i^j * F_j$$

and to consider the equation modulo terms $*$-proportional to F_i. This indeed gives off-shell CHS on the boundary (can be immediately guessed by thinking in trems of constrained systems).

Analogous trick one can try to employ in the case where $V_0^2 = -1$. Although the full system is formally consistent, in linearisation around AdS_{d+1}-background the gauge symmetry:

$$\delta f_i = \lambda_i^j F_j^0 = -\lambda_i^3 + \dots$$

allows to gauge-away everything if one works in the natural local field theory functional space (formal series in Y and polynomials in P). This is likely related to another incarnation of the locality problem in the Vasileiv theory.

Another interpretation of this system: it an FG-like ambient lift of the boundaru off-shell CHS theory. Alternatove holographic reconstruction? (cf. talks by *Ponomarev, Taronna*).

Indeed, in the parent formalism $V_0^2 = 0 \to V_0^2 = -1$ corresponds to going from the boundary to the bulk *Bekaert, MG (2012)*.

It is not clear if such functionall class exists at the full nonlinear level but something can still be done certain backgrounds.

Consider a background where

$$F_1 = \frac{1}{2}P^2\,, \qquad F_2 = (Y^A + V_0^A)P_A \qquad F_3 = -\frac{1}{2}(Y + V_0)^2\,,$$

and do not asume A to be AdS-flat connection. Instead, one can take A to be a generic flat HS connection written in terms of $P, Y + V_0$.

Functional class: polynomials in P, formal series in Y such that

$$(\partial_Y \cdot \partial_Y)^l \phi = 0$$

Then there is a twisted traceless projector Π':

$$\phi = \phi_0 + (Y+V_0)^2\phi_{10} + (Y+V)\cdot P\phi_{11} + \dots\,, \qquad \phi_{\dots} \text{– totally traceless}$$

$$\Pi'\phi = \phi_0$$

In this class $\Pi' a = \Pi' f_i$ is a legitimate gauge condition.

In terms of twisted-traceless a, f_i the equations read as

$$da + \Pi'([A_0, a]) = 0\,, \qquad da + \Pi'([A_0, f_1] + P \cdot \partial_Y a) = 0$$

$f_{2,3}$ are already gauged away.

This can be reduced to unfolded form (e.g. as in *Barnich, MG (2006)*). It should have the structure:

$$d\bar{a} + \Pi'([A_0, \bar{a}] = \mu(A_0, A_0, C) \qquad \text{note: } A_0 \in \text{HS algebra}$$

$P \cdot \partial_Y \bar{a} = 0$ and C parametrises the quotiet $f_1 \sim f_1 + P \cdot \partial_Y \epsilon$.

Recent work by *Sharapov, Skvortsov* shows that such μ is a Hochshild cocycle of the HS algebra and it fully determines the *Vasiliev* theory (the deformation is unobstructed due to absence of higher cohomology). It follows, the simple $sp(2)$-system contains nearly all the information of the Vasiliev theory.

Conclusions

- The HS extension of FG ambient metric construction gives a more geometric framework to HS theory. Proper langauge for HS geometry?

- Bulk/bondary relation is implemented manifestly thanks to the Ambient space formalism. Nonlinear CHS fields are reporoduced on the boundary. Classical version of holographic reconsruction?

- Unifies metric-like and frame-like formalism. In particular, F_1 is an ambient version of the meric-like HS field.

- Likely to provide a framework for studying nonlocality issue at more invariant level.