On current contribution to Fronsdal equations

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Higher-Spin Theory and Holography-6

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Nonlinear HS Equations

- Nonlinear higher-spin equations [Vasiliev '92]

$$\mathrm{d}W + W * \wedge W = -i\theta_{\alpha} \wedge \theta^{\alpha} \left(1 + \eta B * \varkappa k\right) - i\bar{\theta}_{\dot{\alpha}} \wedge \bar{\theta}^{\dot{\alpha}} \left(1 + \bar{\eta} B * \bar{\varkappa}\bar{k}\right),$$
$$\mathrm{d}B + W * B - B * W = 0.$$

- Master fields: $W = W\left(Z; Y|K|x|\theta^{A}, dx^{\underline{m}}\right), \quad B = B\left(Z; Y|K|x\right).$
- Star-product:

$$(f * g)(Z, Y) = \int d^4U d^4V e^{iU_A V^A} f(Z+U, Y+U) g(Z-V, Y+V).$$

- Inner Klein operators:

$$\varkappa := \exp\left(iz_{\alpha}y^{\alpha}\right), \quad \bar{\varkappa} := \exp\left(i\bar{z}_{\dot{\alpha}}\bar{y}^{\dot{\alpha}}\right).$$

$$\varkappa * \varkappa = 1, \quad \varkappa * f(z^{\alpha}; y^{\alpha}) = f(-z^{\alpha}; -y^{\alpha}) * \varkappa.$$

- Outer Klein operators k and $\bar{k}$:

$$kk = 1, \quad kf(z^{\alpha}; y^{\alpha}; \theta^{\alpha}) = f(-z^{\alpha}; -y^{\alpha}; -\theta^{\alpha})k.$$

Perturbation theory

- Vacuum:

$$B = 0, \quad W_0 = \Omega_0^{AB} Y_A Y_B + Z^A \theta_A.$$

$$\Omega_0^{AB} Y_A Y_B = -\frac{i}{4}\left(\omega_L^{\alpha\beta} y_\alpha y_\beta + \bar{\omega}_L^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} + 2h^{\alpha\dot{\beta}} y_\alpha \bar{y}_{\dot{\beta}}\right).$$

- Linear order of dynamical sector:

$$\begin{aligned} \mathcal{D}_{ad}\omega\,(Y|K|x) &= L\,(C)\,, \\ \mathcal{D}_{tw}C\,(Y|K|x) &= 0, \end{aligned}$$

where

$$L\,(C) := -\frac{i\eta}{4} h_\beta{}^{\dot{\alpha}} h^{\beta\dot{\alpha}} \bar{\partial}_{\dot{\alpha}} \bar{\partial}_{\dot{\beta}} C\,(0, \bar{y}|K|x)\,k - \frac{i\bar{\eta}}{4} h^\alpha{}_{\dot{\beta}} h^{\alpha\dot{\beta}} \partial_\alpha \partial_\beta C\,(y, 0|K|x)\,\bar{k},$$

$$\mathcal{D}_{ad} f := D^L f + h^{\alpha\dot{\beta}}\left(y_\alpha \bar{\partial}_{\dot{\beta}} + \bar{y}_{\dot{\beta}} \partial_\alpha\right) f,$$

$$\mathcal{D}_{tw} f := D^L f - i h^{\alpha\dot{\beta}}\left(y_\alpha \bar{y}_{\dot{\beta}} - \partial_\alpha \bar{\partial}_{\dot{\beta}}\right) f,$$

$$D^L f := \mathrm{d}f + \left(\omega_L^{\alpha\beta} y_\alpha \partial_\beta + \bar{\omega}_L^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{\partial}_{\dot{\beta}}\right) f.$$

Restoring Fronsdal fields

- We expand fields of the theory as

$$\omega = \sum_{n,m=0}^{\infty} \omega_{n,m}, \quad C = \sum_{n,m=0}^{\infty} C_{n,m},$$

where $f_{n,m} = f_{\alpha(n),\dot{\alpha}(m)} \left(y^{\alpha}\right)^{n} \left(\bar{y}^{\dot{\alpha}}\right)^{m}$.

- We expand all 1-forms in terms of vierbein

$$D^{L} =: h^{\alpha\dot{\beta}} D_{\alpha\dot{\beta}}, \quad \omega_{n,m} =: h^{\alpha\dot{\beta}} \omega_{n,m|\alpha\dot{\beta}}.$$

- Then linearized equations are rewritten as

$$\begin{aligned}
D^{\beta}{}_{\dot{\alpha}} \omega_{n,\bar{n}|\beta\dot{\alpha}} &= -y^{\beta} \bar{\partial}_{\dot{\alpha}} \omega_{n-1,\bar{n}+1|\beta\dot{\alpha}} - \partial^{\beta} \bar{y}_{\dot{\alpha}} \omega_{n+1,\bar{n}-1|\beta\dot{\alpha}} + \frac{i}{2} \eta \delta_{n,0} \bar{\partial}_{\dot{\alpha}} \bar{\partial}_{\dot{\alpha}} C_{0,\bar{n}+2} k, \\
D_{\alpha}{}^{\dot{\beta}} \omega_{n,\bar{n}|\alpha\dot{\beta}} &= -y_{\alpha} \bar{\partial}^{\dot{\beta}} \omega_{n-1,\bar{n}+1|\alpha\dot{\beta}} - \partial_{\alpha} \bar{y}^{\dot{\beta}} \omega_{n+1,\bar{n}-1|\beta\dot{\alpha}} + \frac{i}{2} \bar{\eta} \delta_{\bar{n},0} \partial_{\alpha} \partial_{\alpha} C_{n+2,0} \bar{k}. \\
D_{\alpha\dot{\beta}} C_{n,m} &= i y_{\alpha} \bar{y}_{\dot{\beta}} C_{n-1,m-1} - \partial_{\alpha} \bar{\partial}_{\dot{\beta}} C_{n+1,m+1}.
\end{aligned}$$

Folding the second order

- We restrict ourselves to totally traceless component of the field $\phi_{n,m} := \omega_{n,m|\alpha\dot{\beta}} y^{\alpha} \bar{y}^{\dot{\beta}}$

$$
\begin{aligned}
\bar{y}^{\dot{\alpha}} D_{\alpha\dot{\alpha}} \partial^{\alpha} \phi_{n,m} &= n \cdot m \phi_{n-1,m+1} + \frac{i}{2} \eta \delta_{n,1} m (m+1) C_{0,m+1} k, \\
y^{\alpha} D_{\alpha\dot{\alpha}} \bar{\partial}^{\dot{\alpha}} \phi_{n,m} &= n \cdot m \phi_{n+1,m-1} + \frac{i}{2} \bar{\eta} \delta_{m,1} n (n+1) C_{m+1,0} \bar{k}, \\
y^{\alpha} D_{\alpha\dot{\alpha}} \bar{y}^{\dot{\alpha}} C_{n,m} &= -i (n+1)(m+1) C_{n+1,m+1}.
\end{aligned}
$$

- From this one finds

$$
\begin{aligned}
C_{s+p,p-s} &= -\frac{2 \cdot i^{p-s+1}}{\bar{\eta} s (s+p)! (p-s)!} \left(y^{\beta} D_{\beta\dot{\beta}} \bar{y}^{\dot{\beta}} \right)^{p-s} \left(y^{\alpha} D_{\alpha\dot{\alpha}} \bar{\partial}^{\dot{\alpha}} \right)^{s} \phi_{s,s} \bar{k}, \\
C_{p-s,s+p} &= -\frac{2 \cdot i^{p-s+1}}{\eta s (s+p)! (p-s)!} \left(y^{\beta} D_{\beta\dot{\beta}} \bar{y}^{\dot{\beta}} \right)^{p-s} \left(\bar{y}^{\dot{\alpha}} D_{\alpha\dot{\alpha}} \partial^{\alpha} \right)^{s} \phi_{s,s} k.
\end{aligned}
$$

$$
\Box \phi_{s,s} + \ldots = -s^2 (s-1) \mathcal{J}_{s,s} + \ldots,
$$

Quadratic corrections

- Quadratic order in canonical form [Gelfond, Vasiliev]:

$$\begin{aligned} \mathcal{D}_{ad}\omega + [\omega,\omega]_* &= L(C) + Q(C,\omega) + \Gamma(J)\,, \\ \mathcal{D}_{tw}C + [\omega, C]_* &= -\mathcal{H}_\eta(J) - \mathcal{H}_{\bar\eta}(J) + \mathcal{D}_{tw}\mathcal{B}(J)\,. \end{aligned}$$

- Current sector ($s \geq s_1 + s_2$):

$$D_{\alpha\dot\beta}\omega_{s-2,s|\alpha}{}^{\dot\beta} = -\bar y_{\dot\beta}\partial_\alpha\omega_{s-1,s-1|\alpha}{}^{\dot\beta} - y_\alpha\bar\partial_{\dot\beta}\omega_{s-3,s+1|\alpha}{}^{\dot\beta} + \partial_\alpha\partial_\alpha\mathcal{J}_{s,s}$$
$$D_{\beta\dot\alpha}\omega_{s,s-2|}{}^{\beta}{}_{\dot\alpha} = -y_\beta\bar\partial_{\dot\alpha}\omega_{s-1,s-1|}{}^{\beta}{}_{\dot\alpha} - \bar y_{\dot\alpha}\partial_\beta\omega_{s+1,s-3|}{}^{\beta}{}_{\dot\alpha} + \bar\partial_{\dot\alpha}\bar\partial_{\dot\alpha}\mathcal{J}_{s,s}$$

where

$$\mathcal{J}_{s,s} = i\eta\bar\eta\frac{(s-2)!}{8\,(2s)!}\sum_{k,m=0}^{s}\frac{(m+k)!\,(2s-m-k)!}{(s-k)!k!\,(s-m!)\,m!}\cdot$$
$$\cdot\left(y^\alpha\partial^1_\alpha\right)^m\left(-y^\beta\partial^2_\beta\right)^{s-m}\left(\bar y^{\dot\alpha}\bar\partial^1_{\dot\alpha}\right)^{s-k}\left(-\bar y^{\dot\beta}\bar\partial^2_{\dot\beta}\right)^k\cdot$$
$$\cdot\left\{\sum_{n=0}^{s}\frac{i^n}{(s+n-1)!}\left(\left(\partial^1_\gamma\partial^{2\gamma}\right)^n + \left(\bar\partial^1_{\dot\gamma}\bar\partial^{2\dot\gamma}\right)^n\right)C\left(Y^1|K|x\right)C\left(Y^2|K|x\right)\right\}\Big|_{Y^1=Y^2=0}.$$

Structure of the current

- With respect to helicity operator $\frac{1}{2}\left(y^{\alpha}\partial_{\alpha} - \bar{y}^{\dot{\alpha}}\bar{\partial}_{\dot{\alpha}}\right)$

$$C = C_+ + C_- + C_0, \qquad \begin{array}{l} C_{n,m} \in C_+, \text{ if } n > m, \\ C_{n,m} \in C_0, \text{ if } n = m, \\ C_{n,m} \in C_-, \text{ if } n < m. \end{array}$$

- Inside of $\mathcal{J}_{s,s}$ terms with equal helicities coupled have $(s + s_1 + s_2)$ derivatives, while terms with opposite helicities have $(s + s_1 + s_2 - 2s_{min})$ derivatives. Then one gets for positive helicities

$$\mathcal{J}_{s,s} \sim \sum_{k,m=0}^{s} \frac{(m+k)!\,(2s-m-k)!}{(s-k)!k!\,(s-m!)\,m!} \cdot$$

$$\cdot \sum_{n=0}^{s} \frac{1}{(s+n-1)!} \left(\partial^1_{\gamma}\partial^{2\gamma}\right)^n C_{s_1+p_1,p_1-s_1}\left(Y^1\right) C_{s_2+p_2,p_2-s_2}\left(Y^2\right) + h.c.$$

where

$$p_1 + p_2 = s + s_1 + s_2, \quad n = s_1 + s_2,$$
$$k = p_2 - s_2, \quad m = p_1 - s_2.$$

Structure of the current

Considering transverse traceless contribution and modulo field redifinitions we have useful identities:

$$D_{\alpha\dot{\alpha}}\phi^{\alpha\ldots\dot{\alpha}\ldots} = 0,$$

$$\ldots(\Box\phi)\ldots\phi = 0,$$

$$\Box(\ldots\phi\ldots\phi) = 0.$$

We can also treat derivatives as commuting (flat limit). All that allows to put long spinor expressions in necessary form, for instance,

$$D_{\alpha\dot{\beta}}\phi_{\gamma}{}^{\dot{\beta}\ldots} = D_{\gamma\dot{\beta}}\phi_{\alpha}{}^{\dot{\beta}\ldots}, \quad D_{\alpha\dot{\beta}}D_{\beta}{}^{\dot{\beta}\ldots} = 0, \quad D_{\alpha\dot{\alpha}}D_{\beta\dot{\beta}} = D_{\beta\dot{\alpha}}D_{\alpha\dot{\beta}}$$

$$D_{\gamma\dot{\gamma}}D_{\alpha\dot{\beta}}\phi_{\beta}{}^{\dot{\beta}\ldots} = D_{\beta\dot{\gamma}}D_{\alpha\dot{\beta}}\phi_{\gamma}{}^{\dot{\beta}\ldots}$$

Currents in spinors

Playing with spinor indices yields

$$\Box\phi_{\mu(s),\dot{\mu}(s)} + \ldots = \frac{i\,(-1)^{s}\,(s_1-1)!\,(s_2-1)!}{2\cdot(s-1)!}\,(-1)^{s}\,(-i)^{-s+s_1+s_2}$$

$$\frac{1}{(s+s_1+s_2-1)!}\sum_{n=0}^{s}\left(\eta^2+\bar{\eta}^2\right)a_n\left[(D_{\mu\dot{\mu}})^{p_1-s_2}\left(D_{\beta\dot{\beta}}\right)^{s_2}\phi^{\alpha(s_1),\dot{\alpha}(s_1)}\right]$$

$$\left[(D_{\mu\dot{\mu}})^{p_2-s_1}\,(D_{\alpha\dot{\alpha}})^{s_1}\,\phi^{\beta(s_2),\dot{\beta}(s_2)}\right]+$$

$$+\left(\eta^2-\bar{\eta}^2\right)a_n\left[(D_{\mu\dot{\mu}})^{p_1-s_2}\left(D_{\beta\dot{\beta}}\right)^{s_2-1}D_{\gamma\dot{\gamma}}\phi^{\alpha(s_1-1),\dot{\alpha}(s_1-1)}{}_{\delta\dot{\delta}}\right]$$

$$\left[(D_{\mu\dot{\mu}})^{p_2-s_1}\,(D_{\alpha\dot{\alpha}})^{s_1-1}\,D^{\delta\dot{\gamma}}\phi^{\gamma\beta(s_2-1),\dot{\delta}\dot{\beta}(s_2-1)}\right]+\ldots$$

Fronsdal equations with currents

- Restoring vector indices, taking into account $Tr\left(\sigma_a\sigma_b\right) = 2\eta_{ab}$ and rescaling fields

$$\Box\phi_{A(s)} + \ldots = \frac{(-1)^s}{(s+s_1+s_2-1)!}2^{\frac{s_1+s_2+s}{2}-1}\cdot$$

$$\cdot\sum_{n=0}^{s}\left(\eta^2+\bar{\eta}^2\right)a_n\left\{(D_A)^n(D_C)^{s_2}\phi^{B(s_1)}\right\}\left\{(-D_A)^{s-n}(D_B)^{s_1}\phi^{C(s_2)}\right\}+$$

$$+\left(\eta^2-\bar{\eta}^2\right)a_n\left\{\epsilon^{MNPQ}(D_A)^n(D_C)^{s_2-1}D_M\phi_N{}^{B(s_1-1)}\right\}\left\{(-D_A)^{s-n}(D_B)^{s_1-1}D_P\phi_Q{}^{C(s_2-1)}\right\}+\ldots$$

where

$$a_n = \frac{s!s!\,(s+s_1-s_2)!\,(s-s_1+s_2)!}{(2s)!\,(s_2-s_1+n)!\,(s+s_1-s_2-n)!\,(s-n)!n!},\quad \sum_{n=0}^{s}a_n = 1.$$

- For minimal A-model ($\eta = \bar{\eta} = 1$, even spins) this gives after integration by parts

$$\Box\phi_{A(s)} + \ldots = \frac{2^{\frac{s_1+s_2+s}{2}-1}}{\Gamma\left(s+s_1+s_2\right)}(D_A)^s(D_C)^{s_2}\phi^{B(s_1)}(D_B)^{s_1}\phi^{C(s_2)}+\ldots$$

that corresponds up to a constant factor [Sleight, Taronna '16] where the vertex was restored from boundary free scalar theory.