On scattering of higher spins in flat space

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D. Ponomarev and AT   arXiv:1603.06273

R. Roiban and AT   arXiv:1701.05773

cf. talks of D. Ponomarev and M. Taronna

Motivation:

- understand properties of theories with infinite number of states:
e.g. consistent massless higher spin theory in AdS (vector dual)
or tensionless limit of string theory in AdS (adjoint dual)

- HS theory in AdS is complicated:
action? locality? consider some limit / simpler case

- HS theory in flat-space ... no-go theorems ...
such theory may exist if relax locality condition?
hidden symmetries?
trivial or nearly S-matrix?

Summary:

• construction of quartic HS interaction vertices
for single tower of massless even spins $s = 0, 2, 4, ...$
using Lorentz-covariant S-matrix approach

• $000s$: minimal choice of 4-vertices required to make
amplitudes on-shell gauge invariant local for $s = 2, 4$ only

• locality may be restored by extending set of fields:
extra tower of (ghost-like) spins $s > 0$ with specific couplings

• indications that extended local action has trivial S-matrix
in agreement with soft limit constraints on S-matrix
from gauge invariance under assumption of locality

• underlying global symmetry of flat-space HS theory?
analogy with conformal extension of Einstein theory:
invariance under conformal HS algebra $\rightarrow$ trivial S-matrix?
contact terms may still be allowed? their interpretation? AdS ?

Plan:

- scattering via massless HS exchanges:
0000 and 000s amplitudes

- constraints from gauge invariance of S-matrix in soft limit

- S-matrix approach to construction of gauge-invariant action:
  non-local 000s 4-vertices

- resolving non-locality by introducing extra tower of states

- conformal off-shell extension:
  Einstein theory and possible HS generalization

Massless higher spins in flat 4d space

- free theory: symmetric double-traceless rank s tensors

$$S = \int d^4x\, \partial^n \phi^{m_1 \ldots m_s} \partial_n \phi_{m_1 \ldots m_s} + \ldots$$

$$\delta \phi_{m_1 \ldots m_s} = \partial_{(m_1} \epsilon_{m_2 \ldots m_s)}, \qquad s = 0, 1, 2, \ldots$$

- cubic interactions with linearized gauge invariance known
- quartic interactions? consistent interacting theory?
- various $s > 2$ “no-go theorems”

e.g. no minimal interactions – no long-range forces

[Weinberg; Cachazo, Benincasa ,...; Bekaert, Boulanger, Sundell 10]

- assumptions? locality of quartic and higher interactions
- demand gauge invariance: which type of non-locality required?
- resolve non-locality introducing new fields?
- then resulting S-matrix is trivial? underlying symmetries?

Why of interest?

• tensionless limit of string theory in flat space?

degenerate ... but well-defined in AdS:

“leading Regge trajectory” – massless tower of higher spins

• massless HS theory in AdS

• consistent non-linear equations known [Vasiliev]

but complicated, many auxiliary fields, so far no action

• action for physical Fronsdal fields

can be reconstructed in principle using AdS/CFT:

match correlators of boundary CFT

[Bekaert,Erdmenger,Ponomarev,Sleight 15; Taronna, Sleight 16,17]

• cubic vertices known; quartic are complicated

issue of locality is subtle / unclear – kernels $f(a\,\partial)$, $\Lambda = 1/a^2$

• flat-space limit of AdS HS theory?

non-local theory for HS tower $s = 0, 1, 2, ...$?

- consistent theory requires

– infinite tower of spins $s = 0, 1, 2, 3, ..., \infty$

– higher derivative (non-minimal) cubic interactions $(s_1 \leqslant s_2 \leqslant s_3)$

$\partial^n \phi_{s_1} \phi_{s_2} \phi_{s_3}, \quad s_2 + s_3 - s_1 \leqslant n \leqslant s_2 + s_3 + s_1$

e.g. l.c. 2-2-2 vertex – $\partial^2, \partial^4, \partial^6$ and 2-3-3 vertex – $\partial^4, ..., \partial^6$

[light-cone: Bengtsson, Bengtsson, Brink; Metsaev;

covariant: Fotopoulos, Tsulaia; Boulanger, Leclerc, Sundell;

Manvelyan, Mkrtchyan, Ruhl; Sagnotti, Taronna, ...]

- Noether procedure: deform $\delta\phi_s = \partial\epsilon_{s-1} + ...$, add 4-vertex,...
- 3-point coupling constants [Metsaev]

$$c_{s_1 s_2 s_3} = g \, \frac{\ell^{s_1+s_2+s_3-1}}{(s_1+s_2+s_3-1)!}$$

- two constants (cf. string th.): g and ℓ= length general structure of action:

$$\frac{1}{g^2}\int d^4x\Big[\sum_s \partial\phi_s\partial\phi_s+\sum \ell^{n-1}\partial^n\phi_{s_1}\phi_{s_2}\phi_{s_3}+\sum \ell^{k-2}\partial^k\phi^4+...\Big]$$

effectively "non-local": no. of ∂ grows with s and no. of ϕ

Aim: find minimal quartic vertex required by gauge invariance

Free higher spin action

- symmetric tensors $\phi_s(x,u)=\phi^{a_1\dots a_s}(x)\,u_{a_1}\dots u_{a_s}$
- Fronsdal action: gauge-inv $\int \phi_s\,\Box\,\phi_s$, 2 d.o.f.

$$S^{(2)}[\phi_s]=\int d^4x\,\phi_s(x,\partial_u)\,\widehat{T}\,[\Box_x-(u\cdot\partial_x)\,\widehat{D}]\,\phi_s(x,u)\Big|_{u=0}$$

$$\widehat{T}=1-\tfrac{1}{4}u^2\partial_u^2,\qquad \widehat{D}\equiv(\partial_x\cdot\partial_u)-\tfrac{1}{2}(u\cdot\partial_x)\partial_u^2$$

- ϕ_s double-traceless $\quad (\partial_u^2)^2 \phi_s(x,u) = 0$
- linearized gauge transformations

$$\delta_s^{(0)} \phi_s(x,u) = (u \cdot \partial_x) \varepsilon_{s-1}(x,u)$$

with traceless parameter $\quad \partial_u^2 \, \varepsilon_{s-1}(x,u) = 0$

- de Donder gauge:

$$\widehat{D} \, \phi_s(x,u) = 0 \qquad \to \qquad \partial^{a_1} \phi_{a_1 \ldots a_s} + \ldots = 0$$

$$S^{(2)}[\phi_s] = s! \int d^4x \; \phi_s(x, \partial_u) \, \widehat{T} \, \Box_x \, \phi_s(x,u) \, \Big|_{u=0}$$

Cubic interaction vertices:

- requiring gauge invariance of combined action

$\delta^{(0)} S^{(3)} + \delta^{(1)} S^{(2)} = 0$ [Manvelyan et al; Sagnotti,Taronna; Joung et al 11]

• traceless-transverse part of cubic vertex $\quad (\partial_{x_{ij}} \equiv \partial_{x_i} - \partial_{x_j})$

$$S^{(3)}[\phi_0, \phi_{s_2}, \phi_{s_3}] = c_{0 s_2 s_3} \int d^d x \Big[(\partial_{u_2} \cdot \partial_{x_{31}})^{s_2} (\partial_{u_3} \cdot \partial_{x_{12}})^{s_3} \times \phi_0(x_1) \phi_{s_2}(x_2, u_2) \phi_{s_3}(x_3, u_3) \Big]_{\substack{u_i=0 \\ x_i = x}}$$

• $c_{s_1 s_2 s_3}$ fixed in l.c. approach [Metsaev 91] $\quad c_{s_1 s_2 s_3} = g \frac{\ell^{s_1+s_2+s_3-1}}{(s_1+s_2+s_3-1)!}$

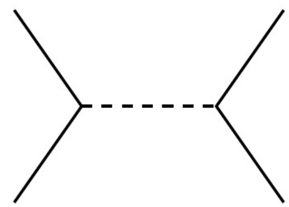
• same $c_{s_1 s_2 s_3}$ for HS in AdS_4 from AdS/CFT [Skvortsov 15; Sleight,Taronna]

HS propagator in de Donder gauge: $\quad \mathcal{D}_s(u, u'; p) = -\frac{i}{p^2} \mathcal{P}_s(u, u')$

$$\mathcal{P}_s(u, u') = \frac{2}{(s!)^2} \Big(\tfrac{1}{2} \sqrt{u^2 u'^2} \Big)^s T_s\Big(\frac{u \cdot u'}{\sqrt{u^2 u'^2}} \Big)$$

$$T_s(z) \equiv \frac{1}{2} \Big[\Big(z + \sqrt{z^2 - 1} \Big)^s + \Big(z - \sqrt{z^2 - 1} \Big)^s \Big]$$

T_s – Chebyshev polynomial of 1st kind



Cubic $0s_2s_3$ vertex : $\quad (p_{ij} \equiv p_i - p_j)$

$$\mathcal{V}(\partial_{u_2}, \partial_{u_3}; p_1, p_2, p_3) = 2ic_{0s_2s_3}(-ip_{31} \cdot \partial_{u_2})^{s_2}(-ip_{12} \cdot \partial_{u_3})^{s_3}$$

Consider scattering of spin 0 via all spin s exchanges

4-scalar scattering amplitude: exchange part

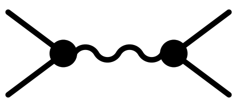
exchange of tower of higher spin fields

[Bekaert, Joung, Mourad 09; Ponomarev, AT 16]

• scalar: $s = 0$ member of HS tower

interactions with even spins only

• s-channel exchange of spin j field

 $\equiv \quad \mathcal{A}^j_{exch}(\mathrm{s},\mathrm{t},\mathrm{u})$

Mandelstam variables $\quad (p_i^2 = p_i'^2 = 0, \quad \mathrm{s}+\mathrm{t}+\mathrm{u}=0)$

$$\mathrm{s} \equiv -(p_1+p_2)^2, \quad \mathrm{t} \equiv -(p_1+p_1')^2, \quad \mathrm{u} \equiv -(p_1+p_2')^2$$

$$\mathcal{A}^j_{exch}(\mathrm{s},\mathrm{t},\mathrm{u}) = -\frac{ic^2_{00j}}{\mathrm{s}} 2^{-j+1} \, (\mathrm{t}+\mathrm{u})^j \;\; T_s\left(\tfrac{\mathrm{t}-\mathrm{u}}{\mathrm{t}+\mathrm{u}}\right)$$

$$\mathcal{A}_{exch}(\mathrm{s},\mathrm{t},\mathrm{u}) = \sum_{j=0,2,4,\dots}^{\infty} \mathcal{A}^j_{exch}(\mathrm{s},\mathrm{t},\mathrm{u})$$

$$\mathcal{A}_{exch}(\mathrm{s},\mathrm{t},\mathrm{u}) = -\frac{i}{\mathrm{s}}\Big[F\big(\sqrt{\mathrm{s}+\mathrm{t}}+\sqrt{\mathrm{t}}\,\big)+F\big(\sqrt{\mathrm{s}+\mathrm{t}}-\sqrt{\mathrm{t}}\,\big)\Big]$$

$$F(z) \equiv \sum_{j=0,2,4,\dots}^{\infty} c_{00\,j}^{2}\,(\tfrac{z^2}{4})^j = \tfrac{1}{8}g^2\,(\ell z)^2\big[I_0(\ell z) - J_0(\ell z)\big]$$

full exchange amplitude

$$\widehat{\mathcal{A}}_{exch}(\mathrm{s},\mathrm{t},\mathrm{u}) = \mathcal{A}_{exch}(\mathrm{s},\mathrm{t},\mathrm{u}) + \mathcal{A}_{exch}(\mathrm{t},\mathrm{s},\mathrm{u}) + \mathcal{A}_{exch}(\mathrm{u},\mathrm{t},\mathrm{s})$$

- Regge limit of exchange part: $\mathrm{t} \to \infty$, s=fixed

$$\widehat{\mathcal{A}}_{exch}(\mathrm{s},\mathrm{t},\mathrm{u}) \sim -\frac{ig^2}{\mathrm{s}}\ell^2\mathrm{t}\,I_0(\ell\sqrt{8\mathrm{t}}) \sim -\frac{ig^2}{\mathrm{s}}\frac{(\ell^2\mathrm{t})^{3/4}}{2^{5/4}\pi^{1/2}}\,e^{\ell\sqrt{8\mathrm{t}}}$$

- fixed angle limit:

$\mathrm{s},\mathrm{t},\mathrm{u} \to \infty$, $\frac{\mathrm{t}}{\mathrm{s}} = -\sin^2\frac{\theta}{2}$, $\frac{\mathrm{u}}{\mathrm{s}} = -\cos^2\frac{\theta}{2}$, θ =fixed

$$\widehat{\mathcal{A}}_{exch}(\mathrm{s},\mathrm{t},\mathrm{u}) \sim ig^2|\mathrm{s}|^{3/4}\,e^{\ell\sqrt{|\mathrm{s}|}\,f(\theta)} \to \infty\,, \qquad f(\theta) > 0$$

• exponential growth: indication of UV divergences in loops
[cf. string theory: Shapiro-Virasoro amplitude is UV-soft]

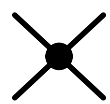
$$A_4 = g^2 \frac{\Gamma(-1-\frac{1}{4}\alpha'\mathrm{s})\Gamma(-1-\frac{1}{4}\alpha'\mathrm{s})\Gamma(-1-\frac{1}{4}\alpha'\mathrm{s})}{\Gamma(2+\frac{1}{4}\alpha'\mathrm{s})\Gamma(2+\frac{1}{4}\alpha'\mathrm{s})\Gamma(2+\frac{1}{4}\alpha'\mathrm{s})}$$

$$\to \quad g^2 |\mathrm{s}|^{-6} (\sin\theta)^{-6} e^{-\alpha'|\mathrm{s}|\, h(\theta)} \to 0$$

$$h(\theta) = -\tfrac{1}{4}\Big(\sin^2\tfrac{\theta}{2}\log\sin^2\tfrac{\theta}{2} - \cos^2\tfrac{\theta}{2}\log\cos^2\tfrac{\theta}{2}\Big) > 0$$

But this is not full amplitude: still to add 4-vertex contribution

0000-vertex contribution



• expected to be effectively non-local: infinite series in ∂^n
• may cancel or "soften" large p behaviour of exchange?
• need extra input to fix 4-scalar vertex in flat-space HS action

• guess from flat limit of AdS action constructed from AdS/CFT

[Bekaert, Erdmenger, Ponomarev, Sleight 2015]: $\nabla \to \partial$

$$S^{(4)}[\phi_0] = g^2 \int d^4x \Big[\sum_{j=0}^{\infty} f_{2j}\big(\Delta_{x_{34}}\big) \left(\partial_{x_{12}} \cdot \partial_{x_{34}}\right)^{2j}$$

$$\times \phi_0(x_1)\phi_0(x_2)\phi_0(x_3)\phi_0(x_4)\Big]_{x_i=x}$$

$$\Delta_{x_{34}} \equiv (\partial_{x_3} + \partial_{x_4})^2, \qquad \partial_{x_{12}} \equiv \partial_{x_1} - \partial_{x_2}$$

• choose $z \to \infty$: $f_{2j}(z) \to c_{2j} \frac{\ell^{4j-2}}{z}$, $c_{2j} = \frac{1}{[(2j-1)!]^2}$

• then contribution to 4-scalar amplitude

$$\sum_{j=0}^{\infty} f_{2j}(\mathrm{s})\,(\mathrm{t}-\mathrm{u})^{2j} = \frac{2\mathrm{t}+\mathrm{s}}{2\mathrm{s}} \left[I_0\big(2\ell\sqrt{2\mathrm{t}+\mathrm{s}}\big) - J_0\big(2\ell\sqrt{2\mathrm{t}+\mathrm{s}}\big)\right]$$

may cancel against the exchange? total amplitude trivial?

S-matrix approach to gauge-invariant interactions

• direct construction of gauge-inv action via Noether procedure: ties construction of action to that of gauge transformations

• more efficient approach: start with S-matrix and demand its on-shell gauge invariance: advantage - only linearized transformations $\delta^{(0)}$ act on physical amplitudes non-linear $\phi\epsilon$ terms in $\delta\phi \sim \partial\epsilon + \phi\,\epsilon + \ldots$, projected out by leg amputation to get S-matrix element

• linearized gauge transformations

$$\delta^{(0)}\phi_s = \partial\epsilon_{s-1} \quad \to \quad \delta\phi_{\mu_1\cdots\mu_s}(p) = p_{(\mu_1}\epsilon_{\mu_2\cdots\mu_s)}(p)$$

• non-trivial case: if S_3 is invariant under linearized g.t. only up to eqs of motion – $p^2 \times \frac{1}{p^2}$ – higher point violation of invariance – add higher vertex to cancel

Example: scalar electrodynamics

$L = \partial^m \phi^* \partial_m \phi + i A^m (\phi^* \partial_m \phi - \phi \partial_m \phi^*) + A^m A_m \phi^* \phi$

$\delta A_m = \partial_m \epsilon, \quad \delta\phi = i\phi\epsilon$

$A(1)\phi(2)\phi(3)A(4)$ scattering amplitude:

$A_m \to \zeta_m(p) e^{ip\cdot x}, \quad p\cdot\zeta = 0$

$$\mathcal{A}_{\rm exch} = \frac{1}{p_{12}^2}\zeta_1\cdot p_2\,\zeta_4\cdot p_3 + \frac{1}{p_{13}^2}\zeta_1\cdot p_3\,\zeta_4\cdot p_2$$

• gauge transformation in leg 1: $\delta\zeta_1 = p_1\epsilon_1, \quad \delta\phi = 0$

$\delta\mathcal{A}_{\rm exch} = (\zeta_4\cdot p_3 + \zeta_4\cdot p_2)\epsilon_1 = -\zeta_4\cdot p_1\,\epsilon_1$

• can be cancelled by adding contact $A^m A_m \phi^* \phi$ vertex

$\mathcal{A}_{\rm cont} = \zeta_1\cdot\zeta_4 \to \ \delta\mathcal{A}_{\rm cont} = p_1\cdot\zeta_4\,\epsilon_1$

• thus 4-point vertex can be found from condition

of linearized gauge invariance of on-shell amplitude

• To get information about structure of possible 4-vertices
consider 0-0-0-s tree-level amplitude:
(i) find exchange contribution
(ii) add general 4-vertex contribution
(iii) impose on-shell gauge invariance w.r.t. spin s leg
(iv) determine "minimal" 4-vertex required by gauge invariance

• Parametrization of $000s$ 4-vertex in momentum space:

$$\mathcal{L}_{000s} = \sum_{k=0}^{s-2} V_{sk}(p_1, p_2, p_3)$$
$$\times\, \phi_0(p_1)\, (2ip_2 \cdot \partial_u)^k \phi_0(p_2)\, (2ip_3 \cdot \partial_u)^{s-k} \phi_0(p_3)\, \phi_j(p_4, u)$$

• Aim: constrain coefficient functions V_{sk}
by demanding that S-matrix element $000s$ is gauge invariant

Gauge-invariance constraints on S-matrix

• soft momentum expansion of massless higher spin amplitudes and gauge invariance constraints: [Low 58; Weinberg 64; Bern et al 14]

• soft limit of massless HS theory with generic 3-couplings

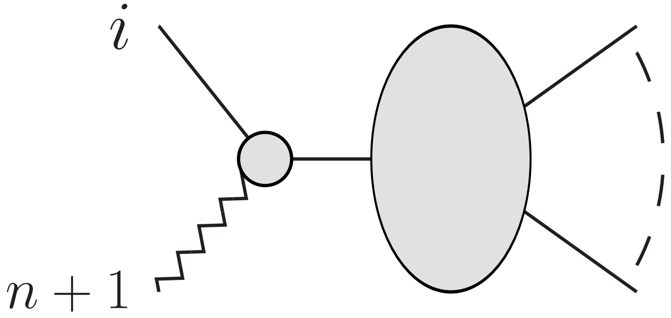
• assume locality: all poles in momentum in amplitudes may only come from on-shell propagators of particles in original action

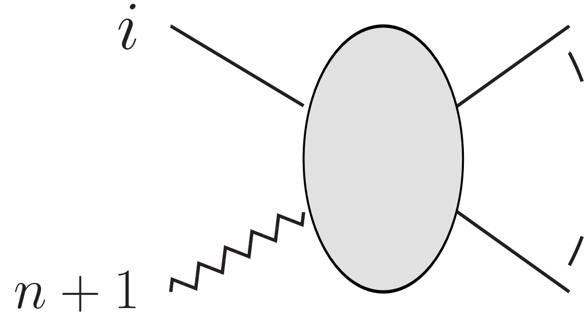
• restrict to leading order of soft momentum expansion: extends [Weinberg 64] to arbitrary couplings of HS [Taronna 11]

Soft momentum expansion of $0...0s$ amplitude

n spin-0 and one spin-s with $p_{n+1} \equiv q \to 0$

two contributions: with pole at $q \to 0$ and without pole





$$\mathcal{A}^{\mu_1\dots\mu_s}(p_1,\dots,p_n,q) = \mathcal{P}^{\mu_1\dots\mu_s}(p_1,\dots,p_n,q) + \mathcal{R}^{\mu_1\dots\mu_s}(p_1,\dots,p_n,q)$$

$$\mathcal{P}^{\mu_1\dots\mu_s} \to \sum_i \sum_{s_i'} \frac{p_i^{\mu_1}\dots p_i^{\mu_s}}{q\cdot p_i} P_{s_i'}(u,u')\left[(p_i-q)\cdot\partial_u\right]^{s_i'} W_{s_i'}(p_i+q,\partial_{u'})$$

$(p_i+q)^2 = 2q\cdot p_i + q^2 \ \to\ 2q\cdot p_i\ , \qquad q\to 0$

$P_s(u,u')$ – projector in spin-s propagator

$W_{s_i'}$ – Green's function with all but i-th leg (p_i+q) on shell

- for $q=0$: $\quad W$ is n-point amplitude

W is then gauge-invariant: $\quad (W_{s_i'})_{q\to 0} = W_{s_i'}(p_i,\partial_{u'})$

$$W_{s_i'}(p_i,\partial_{u'})P_{s_i'}(p_i,u')=0\,,\qquad s_i'\neq 0$$

$$W_{s_i'}(p_i,\partial_{u'})(p_i\cdot u')^k P_{s_i'-k}(p_i,u')=0\,,\qquad k=1,\dots s_i'$$

- gauge invariance of full amplitude requires for any q

$$q_{\mu_s}\mathcal{A}^{\mu_1\dots\mu_s}(p_1,\dots,p_n,q)=0$$

leading term in $q\to 0$:

$$\sum_i\sum_{s_i'}p_i^{\mu_1}\dots p_i^{\mu_{s-1}}\;s_i'!\,W_{s_i'}(p_i,\partial_{u'})P_{s_i'}(p_i,u')=0$$

- assumed locality: dropped $\mathcal{R}$-term that has no poles in q
- gauge inv of $W_{s_i'}(q=0)$: only terms with $s_i'=0$ non-zero
- left with W_0 = $\mathcal{A}^{0\dots0}(p_1,\dots,p_n)$

$$\mathcal{A}^{0\dots0}(p_1,\dots,p_n)\sum_i p_i^{\mu_1}\dots p_i^{\mu_{s-1}}=0$$

• as $\sum_i p_i^{\mu_1} \dots p_i^{\mu_{s-1}}$ does not, in general, vanish if $s > 2$:

$$\mathcal{A}^{0\dots 0}(p_1, \dots, p_n) = 0$$

local action $\to$ scattering amplitude =0 [Weinberg]

Soft momentum expansion of $s_1 \dots s_n s$ amplitude

again $\mathcal{A} = \mathcal{P} + \mathcal{R}$, for $q \to 0$

$$\mathcal{P}^{\mu_1 \dots \mu_s}(p_1, \dots, p_n, q) \to \sum_{i, s'_i} V^{\mu_1 \dots \mu_s}_{s, s_i, s'_i}(q, p_i, \partial_u) \frac{P_{s'_i}(u, u')}{q \cdot p_i} W_{s'_i}(p_i + q, \partial_{u'})$$

$W_{s'_i}$: all but the i-th leg $(q + p_i)$ on shell

for $q = 0$ subject to gauge-invariance constraints

3-vertices $V^{\mu_1 \dots \mu_s}_{s, s_i, s'_i}(q, p_i, \partial_u)$ gauge inv on shell [Manvelyan et al]

non-trivial contribution to spin s gauge inv constraint leading order at $q \to 0$: using explicit form of vertices

$$0 = \sum_i c_{ss_is_i} \frac{1}{s_i!} (u_q \cdot p_i)^{s-1} \phi_{s_i}(p_i, \partial_u^{s_i}) \, W_{s'_i}(p_i, \partial_{u'}) \, P_{s'_i}(u, u')$$

$$= \mathcal{A}^{s_1, \ldots, s_n}(p_1, \ldots, p_n) \sum_i c_{ss_is_i} (u_q \cdot p_i)^{s-1}$$

• $s = 2$: $c_{2s_is_i}$ must be same for all s_i (can use $\sum_k p_k = 0$)

– spin 2 coupling must be universal

• $s > 2$: sum cannot vanish for generic on-shell momenta

• thus gauge invariance requires that either $\mathcal{A}^{s_1 \ldots s_n} = 0$

or constraint on coupling consts: $c_{ss_is_i} = 0 \,, \quad s_i < s$

– no cubic diagonal coupling of spin-s with $s_i < s$ fields

• $0...0s$ amplitude as special case:
if $c_{s00} \neq 0$ and assume locality then $\mathcal{A}_n^{0...0} = 0$
$n = 3$: trivially absent
$n = 4$: vanishing comes from constraint on 5-point $0000s$

• assumed locality (no massless poles) of vertices in action:
i.e. may still get gauge-inv S-matrix in a non-local theory

• if manage to recover locality
(adding extra fields, relaxing unitarity)
but preserving gauge invariance
then total amplitude should still vanish

• gauge invariance itself is not enough to fix S-matrix
locality is standard (strong) extra assumption – implying S=1

0-0-0-s exchange amplitude: [Roiban, AT 17]

• no constraint from soft limit: $\mathcal{A}_3^{000} \equiv 0$

need to go beyond soft limit

• use 0-0-s' and 0-s'-s: $\phi_s \to \zeta_s(p)\, e^{ip\cdot x}$

$\zeta_s(p,q^s) \equiv \zeta_{m_1\ldots m_s}(p)\, q^{m_1}\ldots q^{m_s}, \quad p_{ij} = p_i \cdot p_j, \quad p_i^2 = 0$

• s-channel:

$$\mathcal{A}_{\rm exch} = -\frac{ig^2}{p_{12}^2} \sum_{s'} \frac{\ell^{2s'+s-2}}{(s'-1)!(s+s'-1)!} (p_{12}^2)^{s'} T_{s'}\big(\tfrac{p_{13}^2 - p_{23}^2}{p_{12}^2}\big)\, \zeta_s(p_4, p_3^s)$$

$$T_s(z) = \tfrac{1}{2}\Big[(z+\sqrt{z^2-1})^s + (z-\sqrt{z^2-1})^s\Big]$$

$$\mathcal{A}_{\rm exch} = -\frac{2ig^2}{p_{12}^2}\Big[F_s(z_+) + F_s(z_-)\Big]\, \zeta_s(p_4, p_3^s)$$

$$F_s(z) = z^{2-s}\Big[I_s(z) - J_s(z)\Big]\,, \qquad z_\pm = \ell\big(\sqrt{p_{13}^2} \pm \sqrt{p_{12}^2 + p_{13}^2}\big)$$

• add t and u channels: full $\mathcal{A}_{\rm exch}$

• impose linearized gauge invariance condition

$\delta\zeta_{m_1\ldots m_s}(p) = p_{(m_1}\epsilon_{m_2\ldots m_s)}$

on full amplitude: $\mathcal{A}_4 = \mathcal{A}_{\rm exch} + \mathcal{A}_{cont}$

$$\delta\mathcal{A}_{\rm exch} = -2sg^2\Big[F_s(z_+) + F_s(z_-)\Big]\,\epsilon_{s-1}(p_4, p_3^{s-1}) + \ldots$$

cancel this against variation of contribution of 0-0-0-s vertex

$\sum_{k=0}^{s/2} V_{sk}(p_1, p_2, p_3)\,\phi_0(p_1)(p_2{\cdot}\partial_u)^k\phi_0(p_2)(p_3{\cdot}\partial_u)^{s-k}\phi_0(p_3)\,\zeta_s(p_4, u)$

$\delta\mathcal{A}_{\rm cont} = sV_{s0}(p_1, p_2, p_3)\,p_{24}^2\,\zeta_{s-1}(p_4, p_2^{s-1}) + \ldots$

• find required 4-point vertex V_{000s}

get “minimal” solution consistent (?) with locality

• gauge-invariance: gives relation of V_{sk} to Bessels in $\mathcal{A}_{\rm exch}$

• local solution for 4-vertex exists only for $s = 2$ and $s = 4$

• $s = 2$:

local 4-vertex required by gauge invariance exists:

$$V_{20} = \frac{g^2}{p_{12}^2}\left(F_2(z_+) + F_2(z_-) - \tfrac{1}{2}\left[p_{13}^2 R_2(p_{13}^2) + \text{cycle}\right]\right)$$

$$R_s(x) \equiv \frac{1}{2x}\left[I_s(\sqrt{-x}) - J_s(\sqrt{-x})\right] \quad x \to 0 \text{ residue of } F_2(x)$$

• particular form of gauge-invariant 0-0-0-2 amplitude:

for special choice of local 4-vertex

$$\mathcal{A} = g^2\left[p_{13}^2 R_2(p_{13}^2) + p_{23}^2 R_2(p_{23}^2) + p_{12}^2 R_2(p_{12}^2)\right] \times \left(\frac{\zeta_2(p_4, p_3^2)}{p_{12}^2} + \frac{\zeta_2(p_4, p_2^2)}{p_{13}^2} + \frac{\zeta_2(p_4, p_1^2)}{p_{23}^2}\right)$$

• still not full amplitude: need to fix possible extra terms

in 4-vertex – requires study of other amplitudes

• $s = 4$:

local 4-vertex $\sim R_4 \sim$ Bessels

particular form of gauge-invariant exch + cont 0004 amplitude:

$$\mathcal{A} = U(p_1,p_2,p_3)\,\zeta_4\big(p_4,(p_{12}^2 p_2 - p_{13}^2 p_3)^4\big) - \frac{ip_{12}^2}{15p_{13}^2}\,\zeta_4(p_4,p_2^4) + \ldots$$

$$U = \Big(\frac{1}{p_{13}^2} + \frac{1}{p_{23}^2}\Big) R_4(p_{12}^2) + \text{ cycle}$$

• $s > 4$: no local 4-vertex needed for gauge invariance exists

[Roiban, AT 17; Taronna 11,17]

• cf. constraint of soft theorem:

if assume locality then gauge invariance of $000ss'$

implies vanishing of $000s$

Minimal required non-local 4-vertex for $s \geqslant 6$

to make $000s$ amplitude gauge invariant

coefficient functions $V_{s0}(p_i)$ should have poles ($\ell = 1$)

$$V_{s0}^{\text{nonloc}} = -\frac{1}{p_{12}} \sum_{l=0}^{s/2-3} \kappa_{sl} \big(p_{13}^{2l+2} + p_{23}^{2l+2} \big)$$

$$\kappa_{sl} = \frac{1}{2^{s/2-1} l!(2l+1)!!(l+\frac{s}{2})!(2l+s+1)!!}\,, \qquad p_1 + p_2 + p_3 + p_4 = 0$$

- 4-vertex in position space $\quad \phi(u) \equiv \phi(x, u)$

$$\mathcal{L}_{000s}^{\text{nonloc}} = g^2 \sum_{l=0}^{s/2-3} \phi_0 \left(\partial^{\mu_1} \ldots \partial^{\mu_{2l+2}} \phi_0 \right) \frac{1}{\Box} \left[\partial_{\mu_1} \ldots \partial_{\mu_{2l+2}} (\partial_u \cdot \partial)^s \phi_0 \right] \phi_s(u)$$

• observe factorization in sum over s: $\quad \mathrm{C}_{sl} \equiv \frac{\sqrt{8}g}{2^{s/2-l}(l+\frac{s}{2})!(2l+s+1)!!}$

$$\sum_{s=6,8,\ldots} \mathcal{L}^{\text{nonloc}}_{000s} = \sum_{l=0}^{\infty} \mathrm{C}_{0l}\, \phi_0 \left(\partial^{\mu_1} \ldots \partial^{\mu_{2l+2}} \phi_0\right)$$

$$\times \frac{1}{\Box} \sum_{s=6+2l}^{\infty} \mathrm{C}_{sl} \left[\partial_{\mu_1} \ldots \partial_{\mu_{2l+2}} (\partial_u \cdot \partial)^s \phi_0\right] \phi_s(u)$$

• suggests that one may eliminate non-locality by introducing additional tower of $s = 2, 4, 6, \ldots$ ghosts-like fields ψ_s

$$\mathcal{L}(\phi, \psi) = -\frac{1}{2} \sum_{l=0}^{\infty} \psi_{2l+2} \Box \psi_{2l+2} - \sum_{l=0}^{\infty} \Big[\mathrm{C}_{0l}\, \phi_0 (\partial \cdot \partial_v)^{2l+2} \phi_0$$

$$+ \sum_{s=2l+6}^{\infty} \mathrm{C}_{sl} \left((\partial_u \cdot \partial)^s (\partial_v \cdot \partial)^{2l+2} \phi_0 \right) \phi_s(u) \Big] \psi_{2l+2}(v)$$

• integrating out ψ_j gives also other non-local terms

$$\mathcal{L}^{\mathrm{nonloc}}_{0000} = \sum_{l=0}^{\infty} (\mathrm{C}_{0l})^2 \; \phi_0 \partial^{\mu_1} \dots \partial^{\mu_{2l+2}} \phi_0 \; \frac{1}{\Box} \; \phi_0 \partial_{\mu_1} \dots \partial_{\mu_{2l+2}} \phi_0$$

$$\mathcal{L}^{\mathrm{nonloc}}_{00s_1s_2} = \sum_{l=0}^{\infty} \mathrm{C}_{s_1 l} \; \mathrm{C}_{s_2 l} \; \big[(\partial_{u_1} \cdot \partial)^{s_1} \partial_{\mu_1} \dots \partial_{\mu_{2l+2}} \phi_0\big] \phi_{j_1}(u_1) \\ \times \frac{1}{\Box} \big[(\partial_{u_2} \cdot \partial)^{s_2} \partial_{\mu_1} \dots \partial_{\mu_{2l+2}} \phi_0\big] \phi_{s_2}(u_2)$$

• assume that these non-local quartic terms are indeed present then extra contact contribution to 0000 amplitude

$$(\mathcal{A}^{\mathrm{exch}}_{\mathrm{s}})_{0000}\Big|_{\mathrm{pole}} = -\frac{2ig^2}{p_{12}} \; \mathrm{s}_{13} \big[I_0(\sqrt{8p_{13}}) - J_0(\sqrt{8p_{13}})\big]$$

$$(\mathcal{A}^{\mathrm{ct}}_{\mathrm{s}})_{0000}\Big|_{\mathrm{pole}} = 4i \sum_{l=0}^{\infty} (\mathrm{C}_{0l})^2 (p_{13})^{2l+2} = 2ig^2 \frac{p_{13}}{p_{12}} \big[I_0(\sqrt{8p_{13}}) - J_0(\sqrt{8p_{13}})\big]$$

• total vanishes – cancellation of s-channel pole in 0000
suggests full 0000 amplitude should vanish ?
• same may expect for $s > 0$ – if add proper non-minimal terms
• if local and gauge-invariant but non-unitary extended action
exists – such theory may have a trivial S-matrix
in agreement with expectations based on soft theorem

• other options? gauge-invariant but non-local HS action?
which are the principles that fix it?
hidden symmetry?

Conformal off-shell extension

• candidate symmetry: higher spin conformal symmetry –
symmetry of conformal higher spins $\int d^4x\, \phi\, \Box^s\, \phi$

• analogy: Weyl gravity and conformal extension of Einstein
$\int d^4x \sqrt{g}\,(R\phi^2 + 6\partial^m\phi\partial_m\phi)$ have same symmetries

• similar conformal extension of Fronsdal ∂^2 theory?
requires tower of auxiliary ghost fields – analogs of ϕ

• if one eliminates (integrates out) ghost fields $\rightarrow$
non-local action with extra gauge symmetry but same S-matrix

• $S_E(h) = \int d^4x \sqrt{g} R$

$h_{mn} = t_{mn} + \frac{1}{4}\eta_{mn}\mathrm{h}\,, \qquad t_{mn} \equiv h_{mn} - \frac{1}{4}\eta_{mn}\mathrm{h}\,, \qquad \mathrm{h} \equiv h^m_m$

• h – unphysical – can be gauged away on shell:
does not appear as asymptotic state in S-matrix

• integrate out h – non-local effective action for t_{mn}
produces same Einstein S-matrix

$$\bar{S}_E(t) = \int d^4x(t\partial^2 t + \partial\partial t\partial^{-2}\partial\partial t + \partial^2 ttt$$
$$+\partial^2 t\partial^{-2}\partial t\partial t + \partial^2 tttt + \partial t\partial t\partial^{-2}\partial t\partial t + ...)$$

cf. $S_W(t) = \int d^4x\sqrt{g}\, C^2 = \int d^4x(t\partial^4 t + \partial^4 ttt + \partial^4 tttt + ...)$

• closed form of such action: [Fradkin, Vilkovisky 75]

$$S' = \int d^4x\sqrt{g}\big(R - \tfrac{1}{6}R\Delta^{-1}R\big)$$

Weyl-invariant off shell extension of Einstein theory

• generalize to HS case: quadratic plus cubic action for
Fronsdal HS fields $\phi_{m_1...m_s}$ subject to double-tracelessness
(i) split into "physical" traceless t_s + "ghost-like" trace h_{s-2}
(ii) integrate out h_{s-2}

• resulting non-local action for t_s should lead to same S-matrix:
analog of conformal off-shell extension of Einstein theory
invariant under (spont broken) conformal HS symmetry?

Integrating out the trace from the Einstein action

$$L_E(h) = \sqrt{g}R = X_1 + X_2 + X_3 + X_4 + \ldots\,,$$
$$X_1 = \partial_m\partial_n h_{mn} - \partial^2\mathrm{h} = \partial_m\partial_n t_{mn} - \tfrac{3}{4}\partial^2\mathrm{h}$$
$$X_2 = \tfrac{3}{4}\partial_k t_{mn}\partial_k t_{mn} - \tfrac{1}{2}\partial_k t_{mn}\partial_n t_{mk} + t_{mn}\partial^2 t_{mn} - \partial_n t_{kn}\partial_m t_{km}$$
$$+ \tfrac{3}{32}(\partial_k\mathrm{h})^2 + \tfrac{1}{4}\partial_m t_{mn}\partial_n\mathrm{h} + \tfrac{1}{2}t_{mn}\partial_m\partial_n\mathrm{h}$$
$$X_3 = \tfrac{3}{4}t_{mn}\partial_m t_{sr}\partial_n t_{sr} + t_{ms}\partial_m t_{nr}\partial_n t_{sr} + \tfrac{1}{2}t_{ns}\partial_m t_{nr}\partial_r t_{sm} + \ldots$$

Solving for h:

$$\bar{L}_E(t) = \bar{L}_E^{(2)}(t) + \bar{L}_E^{(3)}(t) + L_E^{(4)}(t) + \ldots$$
$$\bar{L}_E^{(2)}(t) = -\tfrac{1}{4}\partial_k t_{mn}\partial_k t_{mn} + \tfrac{1}{2}\partial_k t_{mk}\partial_n t_{mn} + \tfrac{1}{6}\partial_m\partial_n t_{mn}\partial^{-2}\partial_k\partial_r t_{kr}$$
$$= \tfrac{1}{2}C_{mnkl}\partial^{-2}C_{mnkl} = \tfrac{1}{4}t_{ab}P^{ab}_{mn}\partial^2 t^{mn}$$
$$P^{ab}_{mn} = P^{a}_{(m}P^{b}_{n)} - \tfrac{1}{3}P^{ab}P_{mn}\,, \qquad P_{mn} = \eta_{mn} - \frac{\partial_m\partial_n}{\partial^2}$$

$$\bar{L}_E^{(3)}(t) = X_3(t) + \tfrac{1}{3}X_1(t)\partial^{-2}X_2(t)\,, \qquad X_n(t) \equiv X_n(t, \mathrm{h} = 0)$$

• in transverse gauge $\partial_m t_{mn} = 0$

$$\bar{X}_1 = 0 \,, \qquad \bar{X}_2 = \tfrac{3}{4}\partial_k t_{mn}\partial_k t_{mn} - \tfrac{1}{2}\partial_k t_{mn}\partial_n t_{mk} + t_{mn}\partial^2 t_{mn} \,,$$

$$\bar{X}_3 = -\tfrac{1}{4}t_{ab}\partial_a t_{mn}\partial_b t_{mn} + t_{ab}\partial_a t_{mn}\partial_n t_{mb} - \tfrac{1}{2}t_{ab}\partial_n t_{ma}\partial_n t_{mb} + \ldots$$

$$\bar{X}_4 = -\tfrac{1}{16}t_{mn}t_{mn}(\partial_r t_{ab}\partial_r t_{ab} - 2\partial_r t_{ab}\partial_b t_{ar}) + \ldots$$

$$\bar{L}_E(t) = -\tfrac{1}{4}\partial_k t_{mn}\partial_k t_{mn} + \bar{X}_3(t) + \bar{X}_4(t) + Y_4 \,, \qquad Y_4 = \tfrac{1}{6}\bar{X}_2\partial^{-2}\bar{X}_2$$

• non-local contribution Y_4 to 4-graviton amplitude

$$Y_4 = \tfrac{1}{6}\Big[\tfrac{3}{8}\partial^2(t_{mn}t_{mn}) - \tfrac{1}{2}\partial_k\partial_n(t_{mn}t_{mk})\Big]\frac{1}{\Box}\Big[\tfrac{3}{8}\partial^2(t_{ab}t_{ab}) - \tfrac{1}{2}\partial_r\partial_b(t_{ab}t_{ar})\Big]$$

• complete 4-graviton amplitude =
t_{mn} exchange $\bar{X}_3\partial^{-2}\bar{X}_3$ + local $\bar{X}_4(t)$ + non-local $Y_4(t)$
is physical and gauge-independent
but split between exchange and contact contributions depends
on (on-shell) gauge or particular choice of polarization tensors

Conformal off-shell extension of Einstein theory

• same $\bar{L}_E(t)$ obtained by integrating out h can be found from Weyl-invariant off-shell extension of Einstein theory

$$S(g,\phi) = S_E(\phi^2 g) = \int d^4x \sqrt{g} \big(R\,\phi^2 + 6\,\partial^m\phi\partial_m\phi\big)$$

invariant under $g'_{mn} = \lambda^2(x) g_{mn}, \ \ \phi' = \lambda^{-1}(x)\,\phi$

• perturbatively equivalent to the Einstein theory if assume ϕ has a non-zero constant vacuum value in flat space

i.e. expansion $g_{mn} = \eta_{mn} + h_{mn}, \ \ \phi = 1 + \varphi$

• if fix the Weyl gauge $\varphi = 0 \to$ Einstein theory

  or if solve for φ in terms of the metric $\to$

non-local "conformal off-shell extension" of Einstein gravity

• gives equivalent S-matrix but has an additional Weyl symm

$$\phi(g) = 1 + \varphi(g)\ , \qquad -\nabla^2\varphi + \tfrac{1}{6}R(1+\varphi) = 0$$

$$\varphi = -\tfrac{1}{6}\Delta^{-1}R\ , \quad \Delta \equiv -\nabla^2 + \tfrac{1}{6}R$$

$$S_c(g) \equiv S(g, \phi(g)) = \int d^4x\sqrt{g}\Big(R - \tfrac{1}{6}R\Delta^{-1}R\Big)$$

• Weyl symmetry – can fix traceless gauge on h_{mn}:
S_c depends only on traceless graviton t_{mn} even off-shell
• resulting action is equivalent to $\bar{S}_E = \int d^4x\bar{L}_E(t)$
found by integrating out h from the Einstein action:
either gauge-fixing $\varphi = 0$ and solving for h
or first gauge-fixing $\mathrm{h} = 0$ and solving for φ:
gives same action for t_{mn}

Higher spin generalization?

• Weyl gravity $\to$ conformal higher spin theory
invariant under conformal higher spin symmetry

• conformal extension of Einstein theory $\to$
2-derivative higher spin generalization?
with extra tower of ghost-like "compensator" fields

• solving for extra tower of fields should give
non-local action with extra higher spin conformal symmetry
depending on "physical" traceless parts t_s of Fronsdal fields ϕ_s

• equivalent action (leading to same S-matrix)
from integrating out traces h_{s-2} of the fields ϕ_s in
massless HS Lagrangian $L = \sum_s \phi_s \partial^2 \phi_s + V_3(\phi) + V_4(\phi) + \ldots$

• kin term in non-local action depends only on traceless t_s
represented in terms of linearized Weyl tensors $C_s \sim \partial^s t_s$
conf HS theory: $L_2 = C_s C_s = t_s \Box^s t_s + ...$
conf Fronsdal: $L_2 = C_s \Box^{1-s} C_s = t_s \Box t_s + ...$
• some analogy with "extended" cubic+ quartic theory from
condition of on-shell gauge invariance:
also has extra "ghost-like" HS fields ψ_j needed for locality
• suggests interpretation of ψ_j as
conformal compensators of conformal off-shell extension
that should not appear as asymptotic states in S-matrix
• this proposal may be explaining possible triviality of
resulting S-matrix on the basis of extra hidden symmetry:
allowed terms in amplitudes then are delta-functions of s,t,u
as in conformal higher spin theory

[Joung, Nakach, AT 15; Beccaria, Nakach, AT 16]

Conclusions

• gauge invariance + locality $\rightarrow$ triviality of S-matrix

• using S-matrix gauge invariance to constrain Lagrangian:
0002 and 0004 amplitudes are gauge invariant for local V_4
but $000s$ with $s > 4$ require non-local 4-vertices

• may be eliminated by extra tower of ghost-like HS fields

• this requires, in particular, additional 0000 vertex that cancels
exchange part of 0000 amplitude
if locality can be restored $\rightarrow$ S-matrix is trivial?

• further tests required – e.g. gauge inv of $00s_1s_2$ amplitude

• analogy with conformal off-shell extension of Einstein theory:
higher symmetry explaining triviality of S-matrix?

• theory with "trivial" S-matrix up to contact δ-function terms?

• is there non-local gauge-invariant HS action? symmetry?

- lessons for AdS where “S-matrix” is known

– given by boundary CFT

is there a flat-space limit? is it trivial?