On the applications of AdS/CFT Quantum Spectral Curve to BFKL spectrum of $\mathcal{N} = 4$ SYM

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Motivation

- Using the methods of the Quantum Spectral Curve for $\mathcal{N} = 4$ SYM (Gromov, Kazakov, Leurent, Volin'13; Gromov, Kazakov, Leurent, Volin'14) analytically continue the scaling dimensions of length-2 operators and reproduce the so-called Pomeron eigenvalue of the BFKL equation with nonzero conformal spin (Kotikov, Lipatov'00).

- Derive the generalization of the Faddeev-Korchemsky Baxter equation for the Lipatov's spin chain (known from the integrability of the gauge theory in the BFKL limit) with nonzero conformal spin.

- Find a way for systematic expansion in the scaling parameter in the BFKL regime and study the Pomeron trajectory by numerical and analytical algorithms of QSC.

High-energy scattering

- In the beginning we are going to briefly describe the meaning of the quantities studied in the context of high energy scattering. The total cross-section $\sigma(s)$ for the high-energy scattering of two colorless particles A and B can be written as (Fadin, Lipatov'98; Kotikov, Lipatov'00)

$$\sigma(s) = \int \frac{d^2q\, d^2q'}{(2\pi)^2 q^2 q'^2} \Phi_A(q)\Phi_B(q') \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \left(\frac{s}{s_0}\right)^{\omega} G_\omega(q, q') ,$$

where $s_0 = |q||q'|$ and $s = 2p_A p_B$.

- For the t-channel partial wave there holds the Bethe-Salpeter equation

$$\omega G_\omega(q, q_1) = \delta^{D-2}(q - q_1) + \int d^{D-2} q_2 K(q, q_2) G_\omega(q_2, q_1) .$$

- It appears to be possible to classify the Pomeron eigenvalues ω of the BFKL kernel K using two quantum numbers: integer n (conformal spin) and real ν

$$\omega = \omega(n, \nu) .$$

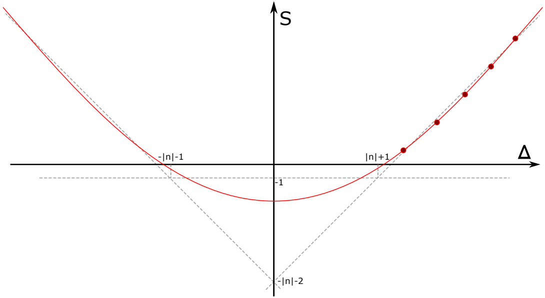
For the phenomenological applications of the BFKL kernel eigenvalues with non-zero conformal spin see (Kepka, Marquet, Royon'10). In (Fadin, Lipatov'98; Kotikov, Lipatov'00) the function ω is used with the different argument $\gamma = 1/2 + i\nu$.

Length-2 operators and BFKL regime in the $\mathcal{N} = 4$ SYM

- We consider important class of length-2 operators

$$\mathrm{tr} Z (D_+)^{S_1} (\partial_\perp)^{S_2} Z + \text{permutations} .$$

- Trajectory $S(\Delta, n)$, where $S = S_1$ and $n = S_2$, corresponding to the length-2 operator tr $Z(D_+)^S (\partial_\perp)^n Z$ with the physical points depicted by the dots



The identification with the high-energy scattering regime is $\omega(n, \nu) = S + 1$, where $\nu = -i\Delta/2$.

- BFKL scaling is determined by: $S \to -1$, $g \to 0$ and $\frac{g^2}{S+1}$ is finite. Leading order BFKL approximation corresponds to resumming all the powers $\left(\frac{g^2}{S+1}\right)^n$.

Algebraic construction. Q-system and QQ-relations

- The $\mathrm{AdS}_5/\mathrm{CFT}_4$ Quantum Spectral Curve (Gromov, Kazakov, Leurent, Volin'13; Gromov, Kazakov, Leurent, Volin'14) gives the generalization of the Baxter equation describing the 1-loop spectrum of twist-2 operators to all loops.

- The AdS/CFT Q-system is formed by 2^8 Q-functions which we denote as $Q_{A|J}(u)$, where $A, J \subset \{1,2,3,4\}$ are two ordered subsets of indices. They satisfy the QQ-relations

$$Q_{A|I}Q_{Aab|I} = Q^+_{Aa|I}Q^-_{Ab|I} - Q^-_{Aa|I}Q^+_{Ab|I},$$
$$Q_{A|I}Q_{A|Iij} = Q^+_{A|Ii}Q^-_{A|Ij} - Q^-_{A|Ii}Q^+_{A|Ij},$$
$$Q_{Aa|I}Q_{A|Ii} = Q^+_{Aa|Ii}Q^-_{A|I} - Q^+_{A|I}Q^-_{Aa|Ii}.$$

  In addition we impose the normalization constraint $Q_{\emptyset|\emptyset} = 1$.

- By applying the QQ-relations we are able to generate the whole Q-system from 8 basic Q-functions: $Q_{a|\emptyset}(u)$ and $Q_{\emptyset|i}(u)$.

Algebraic construction. Hodge and H-symmetry

- By imposing the quantum unimodularity condition $Q_{1234|1234} = 1$ and using the Hodge duality

$$Q_{a_1,\dots,a_n|i_1,\dots,i_m} \leftrightarrow Q^{a_1,\dots,a_n|i_1,\dots,i_m} \equiv$$
$$\equiv (-1)^{(4-n)m} \epsilon^{b_{n+1}\dots b_4 a_1 \dots a_n} \epsilon^{j_{m+1}\dots j_4 i_1 \dots i_m} Q_{b_{n+1},\dots,b_4|j_{m+1},\dots,j_4}$$

we obtain the Hodge dual Q-system with the upper indices, which satisfies the same QQ-relations.

- It also true that

$$Q^{a|i} Q_{a|j} = -\delta^i_j \,, \quad Q^{a|i} Q_{b|i} = -\delta^a_b$$

and

$$Q^{a|\emptyset} = (Q^{a|i})^+ Q_{\emptyset|i} \,, \quad Q^{\emptyset|i} = (Q^{a|i})^+ Q_{a|\emptyset} \,.$$

- The quantum unimodularity condition leads us to the following constraints for the Q-functions

$$Q_{a|\emptyset} Q^{a|\emptyset} = 0 \,, \quad Q_{\emptyset|i} Q^{\emptyset|i} = 0 \,.$$

- QQ-relations are also invariant with respect to the H-transformations

$$Q_{a|\emptyset} \to (H_B)_a^c Q_{c|\emptyset} \,, \quad Q^{a|\emptyset} \to (H_B^{-1})_c^a Q^{c|\emptyset} \,,$$
$$Q_{\emptyset|i} \to (H_F)_i^j Q_{\emptyset|j} \,, \quad Q^{\emptyset|i} \to (H_F^{-1})_j^i Q^{\emptyset|j} \,,$$

where H_B and H_F are periodic matrices and $\det H_B \det H_F = 1$.

Algebraic construction. 4th order Baxter equation

- As a consequence of the QQ-relations, Q-functions with one index are related through the following 4th order finite-difference Baxter equation

$$0 = Q_{\emptyset|j}^{[+4]} D_0 - Q_{\emptyset|j}^{[+2]} \left[D_1 - Q_{a|\emptyset}^{[+2]} Q^{a|\emptyset[+4]} D_0 \right] + \\ + \frac{1}{2} Q_{\emptyset|j} \left[D_2 - Q_{a|\emptyset} Q^{a|\emptyset[+4]} D_0 + Q_{a|\emptyset} Q^{a|\emptyset[+2]} D_1 \right] - \\ - Q_{\emptyset|i}^{[-2]} \left[\bar{D}_1 + Q_{a|\emptyset}^{[-2]} Q^{a|\emptyset[-4]} \bar{D}_0 \right] + Q_{\emptyset|i}^{[-4]} = 0 \,,$$

where

$$D_0 = \det \begin{pmatrix} Q^{1|\emptyset[+2]} & \dots & Q^{4|\emptyset[+2]} \\ Q^{1|\emptyset} & \dots & Q^{4|\emptyset} \\ Q^{1|\emptyset[-2]} & \dots & Q^{4|\emptyset[-2]} \\ Q^{1|\emptyset[-4]} & \dots & Q^{4|\emptyset[-4]} \end{pmatrix} \,, \quad D_1 = \det_{1\leqslant i,j\leqslant 4} Q^{\emptyset|j[4-2i+2\delta_{i,2}]} \,,$$

$$D_2 = \det_{1\leqslant i,j\leqslant 4} Q^{\emptyset|j[4-2i+2\delta_{i,1}+\delta_{i,2}]} \,,$$

$$\bar{D}_k = \det_{1\leqslant i,j\leqslant 4} Q^{\emptyset|j[-4+2i-2\delta_{i,k+1}]} \,, \; k = 0, 1 \,.$$

- After the exchange of the lower and upper indices we obtain the same equation for $Q^{\emptyset|j}$. The four solutions of each equation allow to find four functions $Q_{\emptyset|j}$ and $Q^{\emptyset|j}$ respectively.

Analytic structure. Asymptotics and analytic properties of the basic Q-functions

- We denote the functions $Q_{a|\emptyset}(u)$, $Q_{a|\emptyset}(u)$, $Q_{\emptyset|i}(u)$ and $Q^{\emptyset|i}(u)$ with prescribed analytical properties as $\mathbf{P}_a(u)$, $\mathbf{P}_a(u)$, $\mathbf{Q}_i(u)$ and $\mathbf{Q}^i(u)$ respectively.
- All the Q-functions including $\mathbf{P}_a$, $\mathbf{P}^a$, $\mathbf{Q}_i$ and $\mathbf{Q}^i$ have the power-like asymptotics at large u

$$\mathbf{P}_a \simeq A_a u^{-\tilde{M}_a}\,, \quad \mathbf{P}^a \simeq A^a u^{\tilde{M}_a - 1}\,, \quad \mathbf{Q}_i \simeq B_i u^{\hat{M}_i - 1}\,, \quad \mathbf{Q}^i \simeq B^i u^{-\hat{M}_i}\,,$$

where

$$\tilde{M}_a = \left\{ \frac{J_{1+2-3}}{2} + 1, \frac{J_{1-2+3}}{2}, -\frac{J_{1-2-3}}{2} + 1, -\frac{J_{1+2+3}}{2} \right\}\,,$$
$$\hat{M}_i = \left\{ \frac{\Delta - S_{1+2}}{2} + 1, \frac{\Delta + S_{1+2}}{2}, -\frac{\Delta + S_{1-2}}{2} + 1, -\frac{\Delta - S_{1-2}}{2} \right\}\,.$$

- As we know from the classical integrability of the dual superstring σ-model (see, for example, Gromov'17), the $\mathbf{P}$- and $\mathbf{Q}$-functions at least have the quadratic branch points at $u = \pm 2g$. Natural assumption about their analytic structure on the defining sheet

Analytic structure. Upper and lower half-plane analytic Q-systems

- The equation for the upper half-plane analytic $\mathcal{Q}_{a|i}$ functions

$$\mathcal{Q}^+_{a|i} - \mathcal{Q}^-_{a|i} = \mathbf{P}_a \mathbf{Q}_i \ , \quad \mathcal{Q}_{a|i} \simeq -i \frac{A_a B_i}{-\tilde{M}_a + \hat{M}_i} u^{-\tilde{M}_a + \hat{M}_i} \ , \quad u \to \infty \ .$$

$\mathcal{Q}_{a|i}$ are the Q-functions from the generated UHPA Q-system.

- Substitution of the asymptotics of $\mathcal{Q}_{a|i}$ allows to find the products of A and B coefficients in terms of the charges for $a_0, i_0 = 1, \ldots, 4$

$$A_{a_0} A^{a_0} = i \frac{\prod\limits_{j=1}^{4} \left(\tilde{M}_{a_0} - \hat{M}_j\right)}{\prod\limits_{\substack{b=1 \\ b \neq a_0}} \left(\tilde{M}_{a_0} - \tilde{M}_b\right)} \ , \quad B_{i_0} B^{i_0} = -i \frac{\prod\limits_{a=1}^{4} \left(\hat{M}_{i_0} - \tilde{M}_a\right)}{\prod\limits_{\substack{j=1 \\ j \neq i_0}}^{4} \left(\hat{M}_{i_0} - \hat{M}_j\right)} \ .$$

- Hodge-dual system is also UHPA. $\mathbf{Q}_i$ and $\mathbf{Q}^i$ coincide with $-\mathcal{Q}^+_{a|i} \mathbf{P}^a$ and $\mathcal{Q}^{a|i+} \mathbf{P}_a$ in the UHP. In the UHP and LHP respectively we have

$$\hat{\mathbf{Q}}_i = \check{\mathbf{Q}}_i \ , \quad \hat{\mathbf{Q}}^i = \check{\mathbf{Q}}^i \ , \quad \mathrm{Im}\, u > 0 \ , \quad \tilde{\hat{\mathbf{Q}}}_i = \check{\mathbf{Q}}_i \ , \quad \tilde{\hat{\mathbf{Q}}}^i = \check{\mathbf{Q}}^i \ , \quad \mathrm{Im}\, u < 0 \ .$$

Analytic structure. Gluing conditions

- Complex conjugation generates the LHPA Q-systems with lower and upper indices

$$\mathcal{Q}_{a_1,\dots,a_n|i_1,\dots,i_m}(u) \to (-1)^{\frac{(m+n)(m+n-1)}{2}} \bar{\mathcal{Q}}_{a_1,\dots,a_n|i_1,\dots,i_m}(u) ,$$
$$\mathcal{Q}^{a_1,\dots,a_n|i_1,\dots,i_m}(u) \to (-1)^{\frac{(m+n)(m+n-1)}{2}} \bar{\mathcal{Q}}^{a_1,\dots,a_n|i_1,\dots,i_m}(u) .$$

- As there is no principal difference between the UHPA and LHPA Q-systems and due to the unitarity of $\mathcal{N}=4$ SYM they are connected by the combination of Hodge and H-symmetry

$$\tilde{\bar{\mathbf{Q}}}^i = M^{ij}\bar{\mathbf{Q}}_j , \quad \tilde{\bar{\mathbf{Q}}}_i = \left(M^{-t}\right)_{ij}\bar{\mathbf{Q}}^j .$$

- By using the analyticity properties of the $\mathbf{Q}$-functions we are able to show that the matrix $M^{ij}(u)$ is i-periodic, analytic and hermitian

$$\bar{M}^{ij}(u) = M^{ji}(u)$$

as a function.

Complex conjugation and parity symmetries

- Due to the determined conjugation properties of the $\mathbf{P}$-functions

$$\bar{\mathbf{P}}_a = C_a^b \mathbf{P}_b \,, \quad \bar{\mathbf{P}}^a = -C_b^a \mathbf{P}^b \,, \quad C = \mathrm{diag}\{1, 1, -1, -1\} \,,$$

$\bar{\mathbf{Q}}_i(u)$ also the solutions to the 4th order Baxter equation. Thus, there exist i-periodic matrices that

$$\bar{\mathbf{Q}}_i(u) = \Omega_i^j(u) \mathbf{Q}_j(u) \,, \quad \Omega_i^j = \bar{\mathcal{Q}}_{a|i}^{-} C_b^a \mathcal{Q}^{b|j-} \,.$$

- For the length-2 operators in question ($J_1 = 2$, $J_2 = J_3 = 0$) the $\mathbf{P}$-functions possess the certain parity

$$\mathbf{P}_a(-u) = (-1)^{a+1} \mathbf{P}_a(u) \,, \quad \mathbf{P}^a(-u) = (-1)^a \mathbf{P}^a(u) \,.$$

Thus, $\mathbf{Q}_i(-u)$ are also solutions to the 4th order Baxer equation and

$$\mathbf{Q}_i(-u) = \Theta_i^j(u) \mathbf{Q}_j(u) \,, \quad \Theta_i^j(u) = (-1)^{a+1} \mathcal{Q}_{a|i}^{-}(-u) \mathcal{Q}^{a|j-}(u) \,.$$

Constraining the gluing matrix

- The matrix $M^{ij}\Omega_j^k$ satisfies the equation

$$M^{ij}\tilde{\Omega}_j^k - M^{ij}\Omega_j^k = -\mathbf{Q}^i\tilde{\mathbf{Q}}^k + \mathbf{Q}^k\tilde{\mathbf{Q}}^i \, ,$$

  and possesses the property of antisymmetry $M^{ij}\Omega_j^k = -M^{kj}\Omega_j^i$.

- Using the matrix $\Theta_i^j(u)$ we are able to introduce another gluing matrix

$$\tilde{\mathbf{Q}}^i(u) = L^{ij}(u)\mathbf{Q}_j(-u) \, , \quad \tilde{\mathbf{Q}}_i = \left(L^{-t}\right)_{ij}(u)\mathbf{Q}^j(-u) \, ,$$

  where

$$L^{il}(u) = M^{ij}(u)\Omega_j^k(u)\Theta_k^l(-u) \, ,$$

  which after going under the cut on the real axis twice gives $L^{ji}(u) = L^{ij}(-u)$.

- Summarizing the gluing conditions and the obtained constraints for the gluing matrix we have

$$\bar{M}^{ji}(u) = M^{ij}(u) \, ,$$
$$M^{ij}(u)\Omega_j^k(u) = -M^{kj}(u)\Omega_j^i(u) \, , \; \left(\Omega^{-1}\right)_i^j(u) = \bar{\Omega}_i^j(u) \, ,$$
$$L^{il}(u) = M^{ij}(u)\Omega_j^k(u)\Theta_k^l(-u) \, ,$$
$$L^{li}(-u) = L^{il}(u) \, , \; \left(\Theta^{-1}\right)_j^k(u) = \Theta_j^k(-u) \, .$$

Constraining matrix. Gluing matrix for integer and non-integer spins

- For the integer spins S_1 and S_2 of the same parity we obtain

$$M^{ij} = \begin{pmatrix} 0 & M^{12} & 0 & 0 \\ \bar{M}^{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & M^{34} \\ 0 & 0 & \bar{M}^{34} & 0 \end{pmatrix}$$

supplemented by the fixed phases of non-zero matrix elements

$$M^{12} = \left|M^{12}\right| e^{i\left(\pm\frac{\pi}{2}+\phi_{B_1}-\phi_{B_2}\right)}, \quad M^{34} = \left|M^{34}\right| e^{i\left(\pm\frac{\pi}{2}+\phi_{B_3}-\phi_{B_4}\right)}.$$

- For non-integer spins S_1 and S_2 we have

$$M^{ij}(u) = \begin{pmatrix} M_1^{11} & M_1^{12} & M_1^{13} & M_1^{14} \\ \bar{M}_1^{12} & 0 & 0 & 0 \\ \bar{M}_1^{13} & 0 & M_1^{33} & M_1^{34} \\ \bar{M}_1^{14} & 0 & \bar{M}_1^{34} & M_1^{44} \end{pmatrix} +$$

$$\begin{pmatrix} 0 & 0 & M_2^{13} & M_2^{14} \\ 0 & 0 & 0 & 0 \\ \bar{M}_2^{13} & 0 & 0 & 0 \\ \bar{M}_2^{14} & 0 & 0 & 0 \end{pmatrix} e^{2\pi u} + \begin{pmatrix} 0 & 0 & M_3^{13} & M_3^{14} \\ 0 & 0 & 0 & 0 \\ \bar{M}_3^{13} & 0 & 0 & 0 \\ \bar{M}_3^{14} & 0 & 0 & 0 \end{pmatrix} e^{-2\pi u},$$

where the matrix elements are determined by

$$M_3^{ij} = -M_2^{ij} e^{i\pi(\hat{M}_j - \hat{M}_i)},$$

$$M_2^{13} = \left|M_2^{13}\right| e^{i\left(\pm\frac{\pi}{2}+\phi_{B_1}-\phi_{B_3}\right)}, \quad M_2^{14} = \left|M_2^{14}\right| e^{i\left(\pm\frac{\pi}{2}+\phi_{B_1}-\phi_{B_4}\right)}.$$

Numerical solution. Method description

- The $\mathbf{P}$-functions have the following form on their defining sheet with one short cut

$$\mathbf{P}_a(u) = x^{-\tilde{M}_a}\left(g^{-\tilde{M}_a} A_a \left(1 + \frac{\delta_{a,4}}{x^2}\right) + \sum_{k=1}^{+\infty} \frac{c_{a,k}}{x^{2k}(u)}\right) ,$$

$$\mathbf{P}^a(u) = x^{\tilde{M}_a - 1}\left(g^{\tilde{M}_a - 1} A^a \left(1 + \frac{\delta_{a,1}}{x^2}\right) + \sum_{k=1}^{+\infty} \frac{c^a_k}{x^{2k}(u)}\right) ,$$

  and satisfy the condition $\mathbf{P}_a \mathbf{P}^a = 0$.
- In the limit $u \to \infty$ in the UHP we have the following expansion of $\mathcal{Q}_{a|i}$

$$\mathcal{Q}_{a|i}(u) \simeq u^{-\tilde{M}_a + \hat{M}_i} \sum_{l=0}^{+\infty} \frac{B_{a|i,2l}}{u^{2l}} , \quad B_{a|j,0} = -i \frac{A_a B_j}{-\tilde{M}_a + \hat{M}_j} .$$

  Using the equation

$$\mathcal{Q}^-_{a|i} = \left(\delta^b_a + \mathbf{P}_a \mathbf{P}^b\right) \mathcal{Q}^+_{a|i}$$

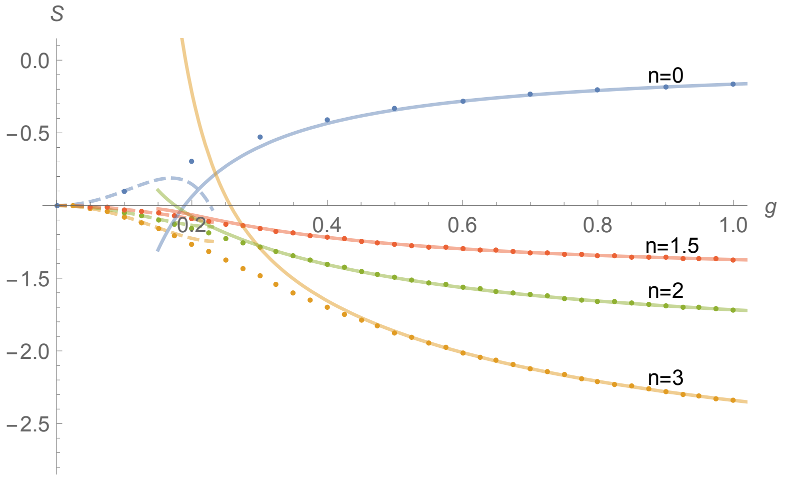
  we find the value of $\mathcal{Q}_{a|i}$ on the real axis.
- The loss function

$$S = \sum_{i,j} |F^i(u_j)|^2 , \quad F^i(u) = \mathcal{Q}^{a|i+}(u) \tilde{\mathbf{P}}_a(u) + M^{ij}(u) \bar{\mathcal{Q}}^-_{b|i}(u) \bar{\mathbf{P}}^b(u) ,$$

  where u_j is a set of points on the interval $[-2g, 2g]$, is minimized by the optimization procedure (Levenberg-Marquardt algorithm).

Numerical solution. Intercept function

- Intercept $S(0, n)$ as the function of the coupling constant g for conformal spins $n = 0$, $n = 3/2$, $n = 2$ and $n = 3$ (dots), weak coupling expansion of the intercept (dashed lines) and strong coupling expansion (continuous lines).



Weak coupling expansion. Asymptotics, symmetries and LO solution

- The length-2 operators are not left-right symmetric. But there is still some symmetry

$$\mathbf{P}^a(n,u) = \chi^{ac}\mathbf{P}_c(-n,u)\,, \quad \mathbf{Q}^i(n,u) = \chi^{ij}\mathbf{Q}_j(-n,u)\,.$$

- The asymptotics are simplified to

$$\begin{aligned} \mathbf{P}_a &\simeq (A_1 u^{-2}, A_2 u^{-1}, A_3, A_4 u)_a\,, \\ \mathbf{Q}_j &\simeq (B_1 u^{\frac{\Delta-n+1-w}{2}}, B_2 u^{\frac{\Delta+n-3+w}{2}}, B_3 u^{\frac{-\Delta+n+1-w}{2}}, B_4 u^{\frac{-\Delta-n-3+w}{2}})_j\,, \end{aligned}$$

where $w = S + 1$.

- After some demanding calculations we get the result for the $\mathbf{P}$-functions

$$\mathbf{P}_1 \simeq \frac{1}{u^2} + \frac{2\Lambda w}{u^4}\,, \quad \mathbf{P}_2 \simeq \frac{1}{u} + \frac{2\Lambda w}{u^3}\,, \quad \mathbf{P}_3 \simeq A_3^{(0)} + A_3^{(1)} w\,,$$

$$\mathbf{P}_4 \simeq A_4^{(0)} u - \frac{i((\Delta^2-1)^2 - 2(\Delta^2+1)n^2 + n^4)}{96u} + $$

$$+ \left(A_4^{(1)} u + \frac{c_{4,1}^{(2)}}{u\Lambda} - \frac{i((\Delta^2-1)^2 - 2(\Delta^2+1)n^2 + n^4)\Lambda}{48u^3}\right) w\,.$$

where $\Lambda = \frac{g^2}{w}$ and

$$c_{4,1}^{(2)} = -\frac{i\Lambda}{24}(\Delta^2 + n^2 + 2((\Delta-n)^2-1)((\Delta+n)^2-1)\Lambda - 1)\,.$$

Weak coupling expansion. Factorization of the 4th order Baxter equation

- Thus, we get the equation for $\mathbf{Q}_1^{(0)}$ and $\mathbf{Q}_3^{(0)}$ in the LO

$$\mathbf{Q}_j^{(0)} \frac{(\Delta - n)^2 - 1 - 8u^2}{4u^2} + \mathbf{Q}_j^{(0)--} + \mathbf{Q}_j^{(0)++} = 0 \,,$$

and for $\mathbf{Q}^{(0)2}$ and $\mathbf{Q}^{(0)4}$ in the LO

$$\mathbf{Q}^{(0)j} \frac{(\Delta + n)^2 - 1 - 8u^2}{4u^2} + \mathbf{Q}^{(0)j--} + \mathbf{Q}^{(0)j++} = 0 \,.$$

Substituting $\mathbf{Q}_j = u^2 Q_j$ and $n = 0$ we immediately see the Baxter equation from (Faddeev, Korchemsky'94) and (Derkachov, Korchemsky, Kotanski, Manashov'01-02).

- In the NLO the 4-th order Baxter equations also factorize and we obtain the following 2nd order Baxter equation

$$\mathbf{Q}^{(1)2,4++} + \mathbf{Q}^{(1)2,4--} + \left(-2 + \frac{(\Delta + n)^2 - 1}{4u^2}\right) \mathbf{Q}^{(1)2,4} =$$
$$= -\frac{i}{2(u+i)} \mathbf{Q}^{(0)2,4++} + \frac{i}{2(u-i)} \mathbf{Q}^{(0)2,4--} + \frac{u^2 - \Lambda(\Delta + n)^2 - 1)}{2u^4} \mathbf{Q}^{(0)2,4} \,.$$

Weak coupling expansion. Gluing conditions and LO BFKL eigenvalue

- Two gluing conditions in the LO in the scaling parameter w

$$\tilde{\mathbf{Q}}^{(0)2} = \bar{\mathcal{M}}_1^{(0)12}\bar{\mathbf{Q}}_1^{(0)} \,,$$
$$\tilde{\mathbf{Q}}^{(0)4} = \bar{\mathcal{M}}_1^{(0)34}\bar{\mathbf{Q}}_3^{(0)} \,.$$

- To find $\mathcal{M}_1^{(0)12}$ and $\mathcal{M}_1^{(0)34}$ we can use the continuity on the cut $\tilde{\mathbf{Q}}^2(0) = \mathbf{Q}^2(0)$ and $\tilde{\mathbf{Q}}^4(0) = \mathbf{Q}^4(0)$. The result is

$$\mathcal{M}_1^{(0)12} = \mathcal{M}_1^{(0)34} = \frac{\cos\frac{\pi(\Delta+n)}{2}}{\cos\frac{\pi(\Delta-n)}{2}}\frac{(\Delta-n)^2-1}{(\Delta+n)^2-1} \,.$$

- After some calculations, we obtain

$$\frac{1}{4\Lambda} = \frac{1}{2}\left(\Psi(\Delta+n)+\Psi(\Delta-n)\right)+\mathcal{O}(g^2) =$$
$$= -\psi\left(\frac{1+n-\Delta}{2}\right) - \psi\left(\frac{1+n+\Delta}{2}\right) + 2\psi(1) + \mathcal{O}(g^2) \,.$$

Weak coupling expansion. BFKL intercept $j(n)$ for general conformal spin n

- Using the binomial harmonic sums

$$\mathbb{S}_{i_1,\dots,i_k}(M) = (-1)^M \sum_{j=1}^{M} (-1)^j \left(\begin{array}{c} M \\ j \end{array} \right) \left(\begin{array}{c} M+j \\ j \end{array} \right) S_{i_1,\dots,i_k}(j) \,.$$

The known intercept functions in the LO and NLO can be expressed in terms of the binomial harmonic sums with the argument $M = (n-1)/2$

$$j_{\rm LO} = 4\mathbb{S}_1 \,, \quad j_{\rm NLO} = 2(\mathbb{S}_{2,1} + \mathbb{S}_3) + \frac{\pi^2}{3}\mathbb{S}_1 \,,$$

and allows to formulate an ansatz for NNLO intercept.

- To calculate the intercept the modified iterative procedure from (Gromov, Levkovich-Maslyuk, Sizov'15) was used. The NNLO intercept is

$$j_{\rm NNLO} = 32(\mathbb{S}_{1,4} - \mathbb{S}_{3,2} - \mathbb{S}_{1,2,2} - \mathbb{S}_{2,2,1} - 2\mathbb{S}_{2,3}) - \frac{16\pi^2}{3}\mathbb{S}_3 - \frac{32\pi^4}{45}\mathbb{S}_1 \,.$$

This result is in complete agreement with (Caron-Huot, Herranen'16). The partial result at the NNNLO order

$$j^{\rm non\text{-}rat.}_{\rm NNNLO}(4k+1) = -\frac{32\pi^2}{3}\left(3\mathbb{S}_{1,4} - 3\mathbb{S}_{2,3} - \mathbb{S}_{3,2} + \mathbb{S}_{1,1,3} - 2\mathbb{S}_{1,2,2} + \mathbb{S}_{2,2,1} - \mathbb{S}_{3,1,1}\right) +$$
$$+\frac{16\pi^4}{15}\left(4\mathbb{S}_3 - \mathbb{S}_{2,1}\right) + \frac{56\pi^6}{135}\mathbb{S}_1 + \frac{32\pi^2\zeta_3}{3}\mathbb{S}_{1,1} + 224\zeta_5\mathbb{S}_{1,1} - 128\zeta_3\left(S_{-3,1} + 2S_{-2,2} -\right.$$
$$-5S_{1,-3} - 15S_{1,3} - 4S_{2,-2} - 12S_{2,2} - 15S_{3,1} - 4S_{-2,1,1} + 2S_{1,-2,1} + 8S_{1,1,-2} +$$
$$\left. +12S_{1,1,2} + 12S_{1,2,1} + 12S_{2,1,1} + S_{-4} + 9S_4\right) \,.$$

Near-BPS all loop expansion. Slope-to-intercept and curvature functions

- Knowing that $\mathbf{P}_a$ and $\mathbf{P}^a$ are $\mathcal{O}(n-1)$ ($\mathcal{O}(\Delta)$) in the LO, we find that $\mathcal{Q}^{(0)}_{a|i}$ to be a constant matrix.
- This allows us to rewrite the equations $\tilde{\mathbf{P}}_a = \mathcal{Q}^{+}_{a|i} M^{ij} \bar{\mathcal{Q}}^{-}_{b|j} \bar{\mathbf{P}}^b$ in the form

$$\tilde{\mathbf{P}}^{(0)}_a = \mathcal{Q}^{(0)+}_{a|i} M^{(0)ij} \bar{\mathcal{Q}}^{(0)-}_{b|j} \bar{\mathbf{P}}^{(0)b}\,, \quad \tilde{\mathbf{P}}^{(1)}_a = \mathcal{Q}^{(0)+}_{a|i} M^{(0)ij} \bar{\mathcal{Q}}^{(0)-}_{b|j} \bar{\mathbf{P}}^{(1)b} +$$
$$+ \left(\mathcal{Q}^{(1)+}_{a|i} M^{(0)ij} \bar{\mathcal{Q}}^{(0)-}_{b|j} + \mathcal{Q}^{(0)+}_{a|i} M^{(0)ij} \bar{\mathcal{Q}}^{(1)-}_{b|j} + \mathcal{Q}^{(0)+}_{a|i} M^{(1)ij} \bar{\mathcal{Q}}^{(0)-}_{b|j} \right) \bar{\mathbf{P}}^{(0)b}\,.$$

- Solving this system, we obtain the slope-to-intercept function

$$\theta(g) = 1 + \frac{I_1(4\pi g) I_2(4\pi g)}{\sum\limits_{k=1}^{+\infty} (-1)^k I_k(4\pi g) I_{k+1}(4\pi g)}\,.$$

- Similar calculations give the curvature function

$$\gamma(g) = \frac{1}{4\pi g^4 I_2^2} \oint\limits_{-2g}^{2g} dv \left(\cosh_-^{v}\, v\,\Gamma[\cosh_-^{u}\, u](v) - \cosh_-^{v}\, v^2 \Gamma[\cosh_-^{u}](v)\right) +$$
$$+ \frac{1}{16\pi g^5 I_2} \oint\limits_{-2g}^{2g} dv \left(\frac{v^3 \Gamma[\cosh_-^{u}](v) - 2v^2 \Gamma[\cosh_-^{u}\, u](v) + v\Gamma[\cosh_-^{u}\, u^2]}{x_v - \frac{1}{x_v}} \right)\,,$$

where

$$\Gamma[h(v)](u) = \oint\limits_{-2g}^{2g} \frac{dv}{2\pi i} \partial_u \log \frac{\Gamma\left[i(u-v)+1\right]}{\Gamma\left[-i(u-v)+1\right]} h(v)\,.$$

Weak and strong coupling expansion of the non-perturbative quantities

- Weak coupling expansion of the slope-to-intercept and curvature functions

$$\theta(g) = -\frac{2\pi^2}{3} g^2 + \frac{4\pi^4}{9} g^4 - \frac{28\pi^6}{135} g^6 + \frac{8\pi^8}{405} g^8 + \mathcal{O}\left(g^{10}\right) ,$$

$$\gamma(g) = 2\zeta_3 g^2 + \left(-\frac{2\pi^2}{3}\zeta_3 - 35\zeta_5\right) g^4 + \left(\frac{16\pi^4}{45}\zeta_3 + \frac{22\pi^2}{3}\zeta_5 + 504\zeta_7\right) g^6 +$$
$$+ \left(-\frac{28\pi^6}{135}\zeta_3 - \frac{8\pi^4}{3}\zeta_5 - 56\pi^2\zeta_7 - 6930\zeta_9\right) g^8 + \mathcal{O}\left(g^{10}\right) .$$

- The strong coupling expansion of the nonperturbative quantities in $\lambda = (4\pi g)^2$ is given by

$$\theta = -1 + \frac{3}{\lambda^{1/2}} - \frac{3}{2\lambda} - \frac{9}{8\lambda^{3/2}} - \frac{9}{4\lambda^2} - \frac{711}{128\lambda^{5/2}} + \mathcal{O}\left(\frac{1}{\lambda^3}\right) ,$$

$$\gamma = \frac{1}{2\lambda^{1/2}} - \frac{1}{4\lambda} - \frac{33}{16\lambda^{3/2}} - \frac{81}{16\lambda^2} - \frac{2265}{256\lambda^{5/2}} +$$
$$+ \frac{1440\zeta_5 - 765}{64\lambda^3} + \frac{207360\zeta_5 - 22545}{2048\lambda^{7/2}} + \mathcal{O}\left(\frac{1}{\lambda^4}\right) .$$

- It is in complete agreement with the corresponding expansion of the intercept found from the numerics

$$S(0,n) = -n + \frac{(n-1)(n+2)}{\lambda^{1/2}} -$$
$$- \frac{(n-1)(n+2)(2n-1)}{2\lambda} + \frac{(n-1)(n+2)(7n^2-9n-1)}{8\lambda^{3/2}} + \mathcal{O}\left(\frac{1}{\lambda^2}\right) .$$

Conclusions and outlook

- We developed a framework for the QSC for both integer and non-integer spins $S_1 = S$ and $S_2 = n$.
- QSC numerical algorithm allowed to calculate S for different values of Δ, n and coupling g.
- We reproduced the dimension of length-2 operator with non-zero conformal spin in the LO of the BFKL regime directly from the QSC.
- Using the iterative procedure, there was obtained the BFKL intercept for arbitrary conformal spin up to NNLO order and partially at NNNLO order.
- We found two new non-perturbative quantities: slope-to-intercept and curvature functions and calculated their weak and strong coupling expansions.
- Find an algorithmic way of generation of any BFKL Pomeron eigenvalue with non-zero conformal spin (NNLO, NNNLO, etc.) on Mathematica program.
- Consider the states with the bigger number of reggeized gluons (Odderon etc.) which means $J_1 \geqslant 3$.
- Incorporate the triple Pomeron vertex into the QSC framework.

Thanks for your attention!