Continuous spin fields of mixed-symmetry type

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Motivation

- Continuous spin fields are massless $m = 0$; the dimensionful parameter $\mu$ (Bargmann, Wigner 1948); infinite number of PDoF.

- Continuous spin dynamics can be defined on the space of fields which is the sum of Fronsdal-like rank-$s$ fields with $s = 0, \ldots, \infty$, similar to the standard interacting HS theories (Fradkin, Vasiliev 1986).

- Action functional on Minkowski space and AdS is the infinite sum of Fronsdal rank-$s$ actions with off-diagonal terms proportional to $\mu$ (Schuster, Toro 2014, Metsaev 2016). The gauge transformations are the standard Fronsdal transformations deformed by Stueckelberg-like terms also proportional to $\mu$. 
Outline

• Group-theoretical description
• Howe duality and higher spin fields
• Equations of motion as constraints
• Triplet formulation
• Metric-like fields and the Schuster-Toro representation
• Light-cone formulation
• Concluding remarks
Group-theoretical description

Continuous spin particles correspond to infinite-dimensional massless UIRs of the Poincare algebra $iso(d - 1, 1)$, induced from infinite-dimensional UIRs of $iso(d - 2)$ subalgebra. Bargmann, Wigner 1948.

Quantum numbers

- a mass $m = 0$
- a continuous spin parameter $\mu \neq 0$
- (half-)integer spin weights $(s_1, \ldots, s_p)$, where $p = \left[ \frac{d - 3}{2} \right]$.

Casimir operators

Generalized Pauli-Lubanski tensors

$$W_{m_1 \ldots m_k} = \epsilon_{m_1 \ldots m_k a_{k+1} \ldots a_d} P^{a_{k+1}} M^{a_{k+2} a_{k+3}} \cdots M^{a_{d-1} a_d}$$

The Pauli-Lubanski tensors covariantly transform under Lorentz subalgebra $o(d - 1, 1)$ and satisfy $[P_a, W_{m_1 \ldots m_k}] = 0$ so that the Casimir operators can be given as

$$C_{2p} = W_{m_1 \ldots m_{p-1}} W^{m_1 \ldots m_{p-1}}$$

For arbitrary representations the Casimir operators can be rather complicated, but in the massless case $C_2 \equiv P^2 = 0$ they are drastically simplified. Denoting $\pi_a = M_{ab} P^b$ we find the general expression

$$C_{2p} \approx [a_{p,0} + a_{p,2} M^2 + \ldots + a_{p,2p-4} M^{2p-4}] \pi_a \pi^a$$

E.g., the quartic Casimir operator is given by $C_4 \sim \pi_a \pi^a$. Then,

- $C_2 = 0$ defines a masslessness
- $C_4$ yields a continuous spin value $\mu^2$ (Brink et al 2002)
- $C_6, C_8, \ldots$ yield spin weights

In other words, a continuous spin representation is characterized by the parameter $\mu$ and $s_1, \ldots, s_p$. The short little algebra $o(d - 3)$. 

Generating function in auxiliary variables

Two types of indices $A^a_I$ running $a = 0, \ldots, d - 1$ and $I = 0, \ldots, n - 1$. We consider polynomials

$$\phi(A) = \sum \phi_{a_1 \ldots a_m; \ldots; c_1 \ldots c_{m-1}} A^{a_1} \ldots A^{a_m} \ldots A^{c_1} \ldots A^{c_{m-1}}$$

Orthogonal algebra $o(d - 1, 1)$:

$$J^{ab} = A^a_I \frac{\partial}{\partial A^b_I} - A^b_I \frac{\partial}{\partial A^a_I}$$

rotations

Symplectic algebra $sp(2n)$:

$$T_{IJ} = A^a_I A^a_J, \quad T^J_I = \frac{1}{2} \{ A^a_I, \frac{\partial}{\partial A^a_J} \}, \quad T^{IJ} = \frac{\partial}{\partial A^a_I} \frac{\partial}{\partial A^a_J}$$

trace creation \hspace{1cm} Young symmetrizer \hspace{1cm} trace annihilation
Howe duality

Finite-dimensional irrep of $o(d - 1, 1)$ algebra

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 \end{array} \]

$s_0$, $s_1$, $\ldots$, $s_{n-1}$

\[ \uparrow \]

Howe duality

\[ \downarrow \]

Highest weight conditions of $sp(2n)$ algebra

\[ T^I_I \phi = s_I \phi \]

\[ T^{IJ} \phi = 0, \quad T^J_I \phi = 0 \quad I > J \]
Introducing Poincare algebra

Remarkably, the auxiliary variables allow us to realize the Poincare algebra as well. Manifest $sp(2n + 2)$ is broken to $sp(2n)$

$$A_0^a \equiv x^a, \quad A_i^a \equiv a_i^a, \quad i = 1, ..., n$$

The Poincare algebra $iso(d - 1, 1)$ basis elements are realized as

$$P_a = \frac{\partial}{\partial x^a}, \quad M_{ab} = x_a \frac{\partial}{\partial x^b} - x_b \frac{\partial}{\partial x^a} + a_{ai} \frac{\partial}{\partial a_i^b} - a_{bi} \frac{\partial}{\partial a_i^a}. $$

Let us introduce notation

$$\Box = \frac{\partial^2}{\partial x^a \partial x^b}, \quad D_i^\dagger = a_i^b \frac{\partial}{\partial x^b}, \quad D_i = \frac{\partial^2}{\partial a_i^b \partial x^b},$$

$$T_{ij}^\dagger = \frac{\partial^2}{\partial a_i^b \partial a_j^b}, \quad T_{ij} = a_i^b a_{bj}, \quad N_i^j = a_i^b \frac{\partial}{\partial a_j^b}, \quad N_i = a_i^b \frac{\partial}{\partial a_i^b}. $$

The above operators form a subalgebra in $sp(2n + 2)$ algebra dual to the Lorentz algebra $o(d - 1, 1)$. The space of formal series in $(x^b, a_i^b)$ is $iso(d - 1, 1) \oplus sp(2n + 2)$ bimodule.
Equations of motion as constraints

Differential constraints

\[ \Box \phi = 0 \, , \quad D^i \phi = 0 \, , \quad i = 1, \ldots, n \, . \]

Algebraic constraints

\[ (T^{ij} + \nu^{ij})\phi = 0 \, , \quad \nu^{ij} = \nu \delta^1_{ij} \, , \quad \nu \in \mathbb{R} \quad i, j = 1, \ldots, n \, , \]
\[ N_i^j \phi = 0 \quad i < j \, , \quad N_i \phi = s_i \phi \, , \quad i, j = 2, \ldots, n \]

Gauge equivalence. The gauge transformations are given by

\[ \delta \phi = \left( D_i^+ + \mu_i \right) \chi^i \, , \quad \mu_i = \mu \delta^1_i \, , \quad \mu \in \mathbb{R} \quad i = 1, \ldots, n \]

Comments:

- At \( \mu, \nu = 0 \) we reproduce the helicity case system (Alkalaev, Grigoriev, Tipunin 2008)
- The constraints are not the highest weight conditions of \( sp(2n + 2) \) algebra: they are typical for the theory of coherent states, where the states are defined as eigenstates of the annihilation operator. State do not diagonalize the spin weight operator \( N_1 \) anymore!
- A functional class: we take formal series in \( a_i^b \) satisfying the additional admissibility condition. A series \( f \) is admissible if its trace decomposition

\[ f = f_0 + f^{ij}_1 T_{ij}^+ + f^{ij}_{2,kl} T_{ij}^+ T_{kl}^+ + \ldots , \quad T^{ij} f_p \cdots = 0 , \]

is such that all coefficients are polynomials of finite order (i.e. for a given \( f \) there exists such \( N \in \mathbb{N} \) that all \( f_r \) are of order not exceeding \( N \)).
Quadratic and quartic Casimir operator

Our formulation involves parameters $\mu, \nu$ and $(n - 1)$ spin weights $s_2, \ldots, s_n$. In $d$ dimensions that allows describing all possible finite-dimensional modules of the short little algebra (Brink et al 2002)

$$o(d - 3) \subset iso(d - 2) \subset iso(d - 1, 1)$$

To characterize $iso(d - 1, 1)$ representations underlying our system we analyze the Casimir operators of the Poincare algebra.

- The quadratic Casimir operator $C_2 = P_a P^a \approx 0$ vanishes on-shell because of $\Box \approx 0$.
- The quartic Casimir operator $C_4 = (M_{ab} P^b)^2$ equals

$$C_4 \phi(x, a) = -D_i^\dagger D_j^\dagger T^{ij} \phi(x, a) \approx \mu^2 \nu \phi(x, a),$$

where we used the differential constraints, trace constrains along with the equivalence relation $\phi \sim \phi + (D^\dagger + \mu) \chi$ with the gauge parameter expressed in terms of the field $\phi$.

Thus, the model propagates continuous spin particles, in which case fixing $\nu = 1$ we identify $\mu$ as the continuous spin parameter. Such a split between deformation parameters $\mu$ and $\nu$ is artificial and only their combination $\mu^2 \nu$ has invariant meaning.
**Triplet formulation**

The triplet BRST operator is

\[
\Omega = c_0 \Box + c_i D^i + (D_i + \mu_i) \frac{\partial}{\partial b_i} - c_i \frac{\partial}{\partial b_i} \frac{\partial}{\partial c_0},
\]

where \( \mu_i = \mu \delta_{i1} \). It is defined on the subspace of \( \Psi = \Psi(x, a|c, b) \) singled out by the BRST extended trace constraints

\[
(\mathcal{T} + \nu)\Psi = 0, \quad \mathcal{T}^\alpha \Psi = 0, \quad \mathcal{T}^{\alpha\beta} \Psi = 0, \quad \alpha, \beta = 2, ..., n
\]

as well as the Young symmetry and the spin weight constraints

\[
\mathcal{N}_\alpha^\beta \Psi = 0 \quad \alpha < \beta, \quad \mathcal{N}_\alpha \Psi = s_\alpha \Psi.
\]

The extended constraints read explicitly as

\[
\mathcal{T}^{ij} = \mathcal{T}^{ij} + \frac{\partial}{\partial c_i} \frac{\partial}{\partial b_j} + \frac{\partial}{\partial c_j} \frac{\partial}{\partial b_i}, \quad \mathcal{N}_\alpha^\beta = \mathcal{N}_\alpha^\beta + b_\alpha \frac{\partial}{\partial b_\beta} + c_\alpha \frac{\partial}{\partial c_\beta},
\]

\[
\mathcal{N}_\alpha = \mathcal{N}_\alpha + b_\alpha \frac{\partial}{\partial b_\alpha} + c_\alpha \frac{\partial}{\partial c_\alpha}, \quad \alpha, \beta = 2, ..., n
\]

Note that the triplet BRST operator is nilpotent \( \Omega^2 = 0 \) on the entire space of unconstrained fields and not only on the subspace singled out by the algebraic constraints.
Representing the ghost number-zero field $\Psi(0)$ as $\Psi(0) = \Phi + c_0 C$ we introduce component fields entering $\Phi = \Phi(x, a|b, c)$ and $C = C(x, a|b, c)$ according to

$$\Phi = \sum_k c_{i_1} \ldots c_{i_k} b_{j_1} \ldots b_{j_k} \Phi^{i_1 \ldots i_k | j_1 \ldots j_k}, \quad C = \sum_k c_{i_1} \ldots c_{i_k} b_{j_1} \ldots b_{j_{k+1}} C^{i_1 \ldots i_k | j_1 \ldots j_{k+1}}.$$

These component fields can be identified as *generalized triplet fields* (Bengtsson 1986). The corresponding gauge transformation reads

$$\delta \Psi(0) = \Omega \Psi(-1),$$

where the ghost number $-1$ parameters $\Psi(-1) = \Lambda + c_0 \Upsilon$ are given by

$$\Lambda = \sum_k c_{i_1} \ldots c_{i_k} b_{j_1} \ldots b_{j_{k+1}} \Lambda^{i_1 \ldots i_k | j_1 \ldots j_{k+1}}, \quad \Upsilon = \sum_k c_{i_1} \ldots c_{i_k} b_{j_1} \ldots b_{j_{k+2}} \Upsilon^{i_1 \ldots i_k | j_1 \ldots j_{k+2}}.$$

The triplet equations of motion for continuous spin fields have the form

$$\Omega \Psi(0) = 0$$

Comments:

- The triplet BRST operator for the continuous spin system differs from the BRST operator for the helicity spin system by adding the term proportional to $\mu$, i.e. $\Omega \rightarrow \Omega + \mu \frac{\partial}{\partial b}$. 
Equivalent dynamical systems

Theory \((\mathcal{H}, \Omega)\):

- \(\mathcal{H}\) – representation space of \(\Omega\), \(\Omega^2 = 0\);
- Equations of motion \(\Omega \phi = 0\), where \(\phi \in \mathcal{H}\).

\[
\mathcal{H} = \mathbb{E} \oplus \mathcal{F} \oplus \mathbb{G}
\]

- \(\mathbb{E}\) – dynamical fields
- \(\mathcal{F}\) – auxiliary fields
- \(\mathbb{G}\) – Stueckelberg fields

Theory \((\mathbb{E}, \hat{\Omega})\):

- \(\mathbb{E}\) – representation space of \(\hat{\Omega}\), \(\hat{\Omega}^2 = 0\);
- Equations of motion \(\hat{\Omega} \psi = 0\), where \(\psi \in \mathbb{E}\).

\((\mathcal{H}, \Omega)\) equivalent \((\mathbb{E}, \hat{\Omega})\)

Additional grading

\[
\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \quad \Omega = \Omega_{-1} + \Omega_0 + \Omega_1 + \ldots
\]

Definition:

\[
\mathbb{E} \oplus \mathbb{G} = \text{Ker}\Omega_{-1}, \quad \mathbb{G} = \text{Im}\Omega_{-1}, \quad \mathbb{E} = \frac{\text{Ker}\Omega_{-1}}{\text{Im}\Omega_{-1}}
\]

See also:

- General approach (Barnich, Grigoriev, Semikhatov, Tipunin '04)
- Light cone DoF, quartets in string theory (Kato, Ogawa '83)
- Unfolded HS formulation (Lopatin, Vasiliev 1988, Shaynkman, Vasiliev '00)
Two reductions of the triplet formulation

- Metric-like formulation (deformed Fronsdal and Labastida formulations)
- Light-cone reduction
Metric-like formulation

Let the additional grading be a homogeneity degree in $c_0$. Then, the triplet BRST operator can be decomposed as $\Omega = \Omega_{-1} + \Omega_0 + \Omega_1$ with

$$\Omega_{-1} = -c_i \frac{\partial}{\partial b_i} \frac{\partial}{\partial c_0}, \quad \Omega_0 = c_i D^i + (D_i^\dagger + \mu_i) \frac{\partial}{\partial b_i}, \quad \Omega_1 = c_0 \Box.$$

The cohomology $H(\Omega_{-1})$ in ghost degree 0 and $-1$ can be explicitly described in terms of the lowest expansion components in ghosts $c_i$ and $b^i$:

$$\Phi = \varphi + \ldots$$
$$\Lambda^i = \chi^i + \ldots$$

The lowest components $\varphi$ and $\chi^i$ satisfy the modified trace conditions

$$T^{(ij)}\varphi = 0, \quad T^{(ij)}\chi^k = 0,$$

where we introduced the notation $T^{ij} \equiv T^{ij} + \nu \delta^{i1} \delta^{j1}$. Young symmetry and spin weight conditions take then the form

$$N_{\alpha\beta} \varphi = 0 \text{ at } \alpha < \beta \quad \text{and} \quad N_{\alpha} \varphi = s_{\alpha} \varphi,$$

and

$$N_{\alpha\beta} \chi^\gamma + \delta^\gamma_{\alpha} \chi^\beta = 0 \text{ at } \alpha < \beta \quad \text{and} \quad N_{\alpha} \chi = s_{\alpha} \chi, \quad N_{\alpha} \chi^\alpha = (s_{\alpha} - 1) \chi^\alpha.$$
Introducing operator $Z$ via $\Omega_{-1} \equiv -\frac{\partial}{\partial c_0} Z$ the original triplet equations $\Omega \Psi^{(0)} = 0$ can be cast into the form

$$\Box \Phi - \Omega_0 C = 0, \quad \Omega_0 \Phi - Z C = 0$$

It follows that $C$ is an auxiliary field and, therefore, using the second equation it can be expressed in terms of $\Omega_0 \Phi$. In other words, $C$ is given by derivatives of $\Phi$, while $\Phi$ itself is reduced to the lowest component $\varphi$. We arrive at

$$\Box \varphi - (D_i^{\dagger} + \mu_i) C^i = 0, \quad D^i \varphi - (D_j^{\dagger} + \mu_j) \Phi^{i|j} - C^i = 0,$$

where the component $\Phi^{i|j}$ can be expressed via $\varphi$ by virtue of the deformed double trace conditions as $\Phi^{i|j} = \frac{1}{2} T^{ij} \varphi$. Eliminating the auxiliary field $C^i$ we finally arrive at the reduced equations of motion

$$\left[ \Box - (D_i^{\dagger} + \mu_i) D^i + \frac{1}{2} (D_i^{\dagger} + \mu_i)(D_j^{\dagger} + \mu_j)(T^{ij} + \nu^{ij}) \right] \varphi = 0,$$

which are invariant with respect to the gauge transformations

$$\delta \varphi = (D_i^{\dagger} + \mu_i) \chi^i .$$

Here, fields and gauge parameters are subject to the algebraic conditions. Note that setting $\mu, \nu = 0$ we reproduce the Labastida formulation.
Scalar continuous spin case

Let us choose \( n = 1 \). In this case, all spin weights vanish \( s_i = 0, \ i = 2, \ldots \). The reduced equations of motion take the form \((\text{Bekaert, Mourad 2005})\)

\[
\Box \varphi - (D^\dagger + \mu) D \varphi + \frac{1}{2} (D^\dagger + \mu)^2 (T + \nu) \varphi = 0, \quad \delta \varphi = D^\dagger \epsilon + \mu \epsilon
\]
supplemented with the deformed trace conditions

\[
(T + \nu)^2 \varphi = 0, \quad (T + \nu) \epsilon = 0
\]

- Note that there are no spin weight conditions in this case. However, the dynamics cannot be restricted to the spin-\(s\) subspace since the deformed trace constraints are incompatible with the spin-\(s\) weight condition \( N \phi = s \phi \).
- Sending both \( \nu \) and \( \mu \) to zero we reproduce a sum of the Fronsdal equations for all integer spins.

The deformed trace conditions can be explicitly solved in terms of tensors subjected to the standard trace conditions

\[
\varphi = \sum_{n,m=0}^{\infty} \beta_{m,n} (T^\dagger)^m \varphi(n), \quad \epsilon = \sum_{n,m=0}^{\infty} \beta_{m,n+1} (T^\dagger)^m \epsilon(n),
\]

where the rank-\(n\) tensors on the right-hand sides satisfy the Fronsdal conditions

\[
T^2 \varphi(n) = 0, \quad T \epsilon(n) = 0,
\]

**Fronsdal basis.** The original \( \varphi \) and \( \epsilon \) are replaced now by infinite collections of Fronsdal (single and double traceless) tensors of ranks running from zero to infinity.
Schuster-Toro representation

It can be explicitly shown that in the Fronsdal basis the metric-like equations take the Schuster-Toro form (\(d = 4\): Schuster, Toro 2014, \(\forall d\): Metsaev 2016)

\[-\Box \varphi(n) + D^\dagger G(n-1) + \mu \left[ G(n) + d_n T^\dagger G(n-2) \right] = 0, \quad n = 0, 1, 2, \ldots, \infty\]

Here,

\[G(n) = A(n) + \mu c_n B(n),\]

with the derivative and algebraic terms combined into

\[A(n) = D\varphi(n+1) - \frac{1}{2} D^\dagger T \varphi(n+1), \quad B(n) = \varphi(n) + a_n T^\dagger T \varphi(n) + b_n T \varphi(n+2),\]

where the coefficients are given by

\[a_n = -\frac{1}{2d + 2n - 8}, \quad b_n = \frac{d + 2n - 2}{2\nu},\]

\[c_n = -\frac{1}{2b_n}, \quad d_n = -\frac{\nu}{(d + 2n - 4)(d + 2n - 6)}.\]

We note that \(A(n)\) and \(B(n)\) as well as \(G(n)\) are traceless. These combinations of fields and their derivatives are convenient to build the double-traceless operator \(G(n)\).

The gauge transformation reads

\[\delta \varphi(n) = D^\dagger \epsilon(n-1) + \mu \left[ \epsilon(n) + d_n T^\dagger \epsilon(n-2) \right].\]

This is the Stueckelberg-like transformation law with three different rank traceless gauge parameters, which is typical for massive higher spin theories (Zinoviev 2001).
Light-cone formulation

The quartet grading is defined by \((a = \pm, m)\)

\[
\deg a_i^\pm = \pm 2, \quad \deg a_i^m = 0, \quad \deg c_0 = 0, \quad \deg c_i = 1, \quad \deg b^i = -1.
\]

The triplet BRST operator decomposes as \(\Omega = \Omega_{-1} + \Omega_0 + \Omega_1 + \Omega_2 + \Omega_3\), where

\[
\Omega_{-1} = p^+ \left( c_i \frac{\partial}{\partial a_i^+} + a_i^- \frac{\partial}{\partial b_i} \right), \quad \Omega_0 = c_0(2p^+ p^- + p_m p^m),
\]

\[
\Omega_1 = c_i p^m \frac{\partial}{\partial a_i^m} + p^+ a_i^- \frac{\partial}{\partial b_i} + \mu \frac{\partial}{\partial b} , \quad \Omega_2 = -c_i \frac{\partial}{\partial b_i} \frac{\partial}{\partial c_0} , \quad \Omega_3 = p^-(c_i \frac{\partial}{\partial a_i^-} + a_i^+ \frac{\partial}{\partial b_i} ) .
\]

We find \(H_0(\Omega_{-1}) = \{ \phi(x|a_i^m) \}\), i.e. these are \(o(d-2)\) tensors. The reduced BRST charge reads

\[
\tilde{\Omega} = c_0(2p^+ p^- + p^m p_m) \equiv c_0 \Box
\]

The light-cone off-shell constraints are given by

\[
(\tilde{T} + \nu)\phi = 0 , \quad \tilde{T}^\alpha \phi = 0 , \quad \tilde{T}^{\alpha\beta} \phi = 0 , \quad \tilde{N}_{\alpha\beta} \phi = 0 , \quad \alpha < \beta , \quad \tilde{N}_\alpha \phi = s_\alpha \phi , \quad \alpha, \beta = 2, \ldots, n , \]

where

\[
\tilde{T}^{ij} = \frac{\partial^2}{\partial a_i^m \partial a_j^m} , \quad \tilde{N}_{\alpha\beta} = a_\alpha^m \frac{\partial}{\partial a_\beta^m} , \quad \tilde{N}_\alpha = a_\alpha^m \frac{\partial}{\partial a_\alpha^m}
\]
Light-cone symmetry

Poincare algebra. The Poincare generators in the light-cone basis split into two groups: kinematical $G_{\text{kin}} = (P^+, P^m, M^+, M^-, M^{mk})$ and dynamical $G_{\text{dyn}} = (P^-, M^- k)$. After quartet reduction both types of generators act in the subspace, $\tilde{G}_{\text{kin}}$ and $\tilde{G}_{\text{dyn}}$. We find out that the reduced kinematical generators $\tilde{G}_{\text{kin}}$ take the standard form, while the reduced dynamical generators $\tilde{G}_{\text{dyn}}$ are given by

$$\tilde{P}^- = -\frac{p^k p_k}{2p^+}, \quad \tilde{M}^{-m} = -\frac{\partial}{\partial p^+} p^m - \frac{\partial}{\partial p_m} \frac{p^k p_k}{2p^+} + \frac{1}{p^+} (S^{mk} p_k + H^m),$$

where $S^{mn}$ and $H^m$ read

$$S^{mn} = a^m_\alpha \frac{\partial}{\partial a^\alpha_n} + a^m_n \frac{\partial}{\partial a^\alpha_m} - (m \leftrightarrow n), \quad H_n = \mu \frac{\partial}{\partial a^n}.$$

The elements $S^{kl}$ and $H^n$ satisfy the $\text{iso}(d-2)$ commutation relations

$$[S^{kl}, S^{ps}] = \delta^{kp} S^{ls} + 3 \text{ terms}, \quad [S^{kl}, H^n] = \delta^{kn} H^l - \delta^{ln} H^k, \quad [H^k, H^l] = 0.$$

Casimir operators. We immediately see that the $\text{iso}(d-2)$ Casimir operators are given by

$$c_2 \equiv H^2 \approx \mu^2 \nu,$$

$$c_4 \equiv H^2 S^2 - 2(HS)^2 \approx \mu^2 \nu \sum_{\alpha=2}^n s_\alpha (s_\alpha + d - 2\alpha - 3),$$

where $H^2 = H^m H_m$, $S^2 = S_{mn} S^{mn}$, $(HS)^m = H_n S^{nm}$. 
Continuous spin-s case

Let us analyze the continuous spin representation labeled by \((s, 0, \ldots, 0)\) in more detail. In this case there are two oscillators \((a, a_1^m)\) and the trace constraints read

\[
(\tilde{T} + \nu)\phi = 0, \quad \tilde{T}^1\phi = 0, \quad \tilde{T}^{11}\phi = 0,
\]

where

\[
\phi = \sum_{p=0}^{\infty} \phi_{m_1 \cdots m_p | n_1 \cdots n_s} a^{m_1} \cdots a^{m_p} a_1^{m_1} \cdots a_1^{n_s},
\]

and the spin weight condition \(\tilde{N}_1\phi = s\phi\) has been taken into account.

Let \(Y(k, l)\) denote a traceless \(o(d-2)\) tensor associated to the Young diagram with \(k\) indices in the first row and \(l\) indices in the second row. Then, the solution is given by

\[
\phi : \bigoplus_{l=0}^{\infty} \bigoplus_{k=s}^{\infty} Y(k, l)
\]

- When \(s = 0\) the above space is an infinite chain of totally symmetric \(o(d-2)\) traceless tensors (Schuster, Toro 2014, Metsaev 2016, 2017).
- For \(s \neq 0\) the space is a light-cone version of the covariant formulation discussed in (Zinoviev 2017).
- Let \(d = 5\): using the the Hodge duality \(Y(k, 1) \sim Y(k, 0)\) and \(Y(k, m) = 0\) at \(m > 1\) we find out the representation space described in (Brink et al 2002, Metsaev 2017), i.e. two infinite chains of traceless \(o(3)\) tensors \(Y(k, 0)\) with \(k = s, s + 1, \ldots, \infty\).
Final comments

Conclusions

• Implementing differential constraints via the BRST operator and imposing algebraic constraints directly we arrive at the triplet formulation for continuous spin. The resulting equations of motion have a simple form even in the general mixed-symmetry case.

• Using the homological reductions of the triplet BRST operator we found the metric-like formulation that generalizes the Schuster-Toro description of the scalar continuous spin fields. On the other hand, the resulting metric-like formulation is the $\mu$-deformation of the Labastida equations.

• Applying the so-called quartet mechanism we can get rid of the unphysical components of the oscillators to obtain the light-cone form of the continuous spin dynamics. In particular, we explicitly built the $iso(d - 2)$ Wigner little algebra and computed its second and fourth Casimir operators.

• There is a functional class so that the gauge symmetry does not kill all PDoF. We demonstrate by performing the light-cone analysis that the system indeed propagates correct degrees of freedom.

Outlooks

• Fermions, SUSY (forthcoming paper with M. Grigoriev and A. Chekmenev)

• Understand group-theoretical meaning of continuous spin fields in AdS

• AdS/CFT correspondence for continuous spin fields...