Homotopy Operators and Locality Theorems in Higher-Spin Equations

O.A.Gelfond

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Introduction and Main Results

After the work of Giombi and Yin (2009) conventional homotopy (Vasiliev 1992) is known to lead to nonlocalities beyond the free field level. (Also Boulanger, Kessel, Skvortsov, Taronna 2015; Vasiliev 2017)

Aim of this talk is to show how to modify homotopy approach to reduce the degree of nonlocality in the sector of zero forms in **all** perturbation orders. This is achieved by proving

Z-dominance Lemma providing a sufficient condition controlling locality of dynamical field equations and
Pfaffian Locality Theorem showing how to choose shifted homotopies to achieve degeneracy of the Pfaffian matrix of derivatives over spinor variables of different fields in higher corrections in the zero-form sector.

For bilinear corrections PLT leads to local results of Vasiliev 2016,2017 for details: talks of Didenko and Koribut.

Nonlinear HS Equations

$$\mathsf{d}_x \mathcal{W} + \mathcal{W} * \mathcal{W} = -i(dZ^A dZ_A + \eta dz^\alpha dz_\alpha B \star k \star \kappa + \bar{\eta} d\bar{z}^{\dot{\alpha}} d\bar{z}_{\dot{\alpha}} B \star \bar{k} \star \bar{\kappa})$$

$$\mathsf{d}_x B + \mathcal{W} * B - B * \mathcal{W} = 0$$

Two-component spinor notations: $Y^A = (y^\alpha, \bar{y}^{\dot{\alpha}})$, $Z^A = (z^\alpha, \bar{z}^{\dot{\alpha}})$, $\alpha, \beta = 1, 2$

HS star product

$$(f * g)(Z; Y) = \int d^4U d^4V f(Z + U; Y + U) g(Z - V; Y + V) \mathrm{e}^{iU_A V^A},$$

$U_A V^A = u^\alpha v^\beta \epsilon_{\alpha\beta} + \bar{u}^{\dot{\alpha}} \bar{v}^{\dot{\beta}} \epsilon_{\dot{\alpha}\dot{\beta}}$, $sp(4)$**-invariant symplectic form** $(\epsilon_{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}})$.

$\mathsf{d}_x = dx^m \frac{\partial}{\partial x^m}$ **space-time de Rham differential**: $\theta^\alpha = dz^\alpha$, $\bar{\theta}^{\dot{\alpha}} = d\bar{z}^{\dot{\alpha}}$,

κ **and** $\bar{\kappa}$ **-inner Klein operators**

$\kappa := \exp(iz_\alpha y^\alpha)$, $\kappa * \kappa = 1$, $\kappa * f(z^\alpha; y^\alpha; dz^\alpha) = f(-z^\alpha; -y^\alpha; dz^\alpha) * \kappa$

k **and** $\bar{k}$ **-outer Klein operators** $k f(z^\alpha; y^\alpha; dz^\alpha) = f(-z^\alpha; -y^\alpha; -dz^\alpha) k$

$\eta = \exp i\phi$ **is a free parameter**

Fields of the Nonlinear System

$$B(Z;Y;K|x)\,,\qquad \mathcal{W}(Z;Y;K|x)=(W_n(Z;Y;K|x)\mathsf{d}x^n\,,\,S_A(Z;Y;K|x)\mathsf{d}Z^A)$$

Zero-forms $B(Z;Y;K|x)$, $K=(k,\bar{k})$,

Spin-s physical fields: Z**-independent part** C **of** B

$$C_s(Y;K|x)=C_s^{1,0}(Y|x)k+C_s^{0,1}(Y|x)\bar{k}$$

$$C_s^{kj}(y,\bar{y}|x)=\frac{1}{2i}\sum_{|m-n|=2s}\frac{1}{m!n!}C^{kj}{}_{\alpha_1\ldots\alpha_n,\dot{\beta}_1\ldots\dot{\beta}_m}(x)y^{\alpha_1}\ldots y^{\alpha_n}\bar{y}^{\dot{\beta}_1}\ldots\bar{y}^{\dot{\beta}_m}$$

One-forms $W(Z;Y;K|x)$, $S(Z;Y;K|x)$,

Spin-s physical fields Z**-independent part** ω **of** W,

$$\omega_s(Y;K|x)=\omega_s^{0,0}(Y|x)+\omega_s^{1,1}(Y|x)k\bar{k}$$

$$\omega_s(y,\bar{y};K|x)=\frac{1}{2i}\sum_{n+m=2(s-1)}\frac{1}{m!n!}\omega_{\alpha_1\ldots\alpha_n,\dot{\beta}_1\ldots\dot{\beta}_m}(k\bar{k}|x)y^{\alpha_1}\ldots y^{\alpha_n}\bar{y}^{\dot{\beta}_1}\ldots\bar{y}^{\dot{\beta}_m}$$

Perturbative Analysis

Vacuum solution $B_0 = 0\,, \qquad S_0 = Z_A dZ^A\,, \qquad W_0 = \frac{1}{2} w^{AB}(x) Y_A Y_B$

$$\mathrm{d}W_0 + W_0 \star W_0 = 0$$

$$[S_0\,, f]_\star = -2i\mathrm{d}_Z f\,, \qquad \mathrm{d}_Z = dZ^A \frac{\partial}{\partial Z^A}$$

$w^{AB}(x)$ **describes** AdS_4.

Generally: arbitrary flat background connection $W_0(Y)$

First-order: $B_1 = C(Y)\,, \quad S = S_0 + S_1\,,$

$$W = W_0(Y) + \omega(Y) + W_0(Y)C(Y)$$

Nontrivial space-time equations on $\omega(Y|x)$ **and** $C(Y|x)$**:**

d_Z**-cohomology**

Reconstruction of Z Variables: Conventional Homotopy

Perturbatively, order$-\ n$ d_Z -dependent equations

$$\mathsf{d}_Z U_n(Z;Y|dZ) = V[U_{j<n}](Z;Y|dZ)\,, \qquad \mathsf{d}_Z V[U_{<n}](Z;Y|dZ) = 0$$

Solution

$$U_n(Z;Y|dZ) = \mathsf{d}^*_{\mathbf{Z}} V[U_{<n}](Z;Y|dZ) + h(Y) + \mathsf{d}_Z \epsilon(Z;Y|dZ)$$

Conventional homotopy operator $\partial = Z^A \frac{\partial}{\partial \theta^A}$ $\rightarrow$ resolution d^*_Z

$$\mathsf{d}^*_{\mathbf{Z}} V(Z;Y|dZ) = Z^A \frac{\partial}{dZ^A} \int_0^1 \frac{dt}{t} V(tZ;Y|tdZ)$$

lead to nonlocalities beyond the free field level.

Shifted Homotopy

An obvious freedom in the definition of homotopy operator

$$Z^A \quad \to \quad Z^A + Q^A \,, \qquad \partial \quad \to \quad \left(Z^A + Q^A\right)\frac{\partial}{\partial\theta^A}\,, \qquad \frac{\partial}{\partial Z^A}(Q^B) = 0$$

$Q_A = \frac{\partial}{\partial Y^A}$ acting on $C(Y)$ is admissible

Let $\Phi^1(Y;K) = \omega(Y;K)$, $\Phi^0 = C(Y;K)$.

$U_n(Y;K)$ contains ordered products

$$\Phi^{\mathbf{a}}(Y;K) = \Phi^{a_1}(Y_1;K)\dots\Phi^{a_n}(Y_n;K)\Big|_{Y_i=Y}\,, \qquad \mathbf{a} = \{a_1,\dots,a_n\} \quad a_i = 0,1.$$

HS equations remain consistent for $\Phi^{a_j}(Y_j;K)$ valued in any associative algebra. $\Rightarrow$Terms with different $\mathbf{a}$-sequences are separately d_z-closed.

$\Rightarrow$ Homotopy operators ∂_Q can be independent for different $\mathbf{a}$-sequences

$$Q^{\mathbf{a}}{}_A = c_0(\mathbf{a})Y_A + \textstyle\sum_j c_j(\mathbf{a})\partial_{jA}\,, \qquad \mathbf{a} = \{a_1,\dots,a_n\}.$$

∂^j_A is the derivative with respect to Y^A_j of the j^{th} factor $\Phi^{a_j}(Y_j;K)$.

c_j can depend on covariantly contracted combinations of ∂_{jA} and Y_A.

Resolution of Identity

Resolution of identity is standard

$$\left\{ \mathsf{d}_Z \, , \Delta_{Q^{\mathbf{a}}} \right\} + \widehat{h}_{Q^{\mathbf{a}}} = Id \, ,$$

Resolution

$$\Delta_{Q^{\mathbf{a}}} J\,(Z;Y;\theta) = \left(Z^A + Q^{\mathbf{a}A} \right) \frac{\partial}{\partial \theta^A} \int_0^1 d\tau \frac{1}{\tau} J\,(\tau Z - (1-\tau) Q^{\mathbf{a}}; Y; \tau\theta) \ ,$$

Cohomology projector

$$\widehat{h}_{Q^{\mathbf{a}}} F(Z,Y) = F(-Q^{\mathbf{a}}, Y) \, .$$

Conventional resolution $\mathsf{d}^*_Z = \Delta_0$

Quadratic Corrections: Old Approach

Nonlocal quadratic corrections to equations in the zero-form sector

$$D^{tw}C + [\omega\,, C]_* + \mathcal{H}_\eta(w, CC) + \mathcal{H}_{\bar\eta}(w, CC) \;= 0$$

$\mathcal{H}_\eta$ contains arbitrary degrees of $\quad \partial_{1\alpha}\partial_2{}^\alpha\bar\partial_{1\dot\alpha}\bar\partial_2{}^{\dot\alpha}$

Local solution by non-local field redefinition $C := C + \Phi_\eta(\mathcal{J}) + \bar\Phi_{\bar\eta}(\mathcal{J})$

$$B^{loc}_{2\eta}(w, CC) \;=\; \mathcal{H}_\eta(w, CC) + D^{tw}\Phi_\eta(CC)$$

Vasiliev (2016)

$$B^{loc}_{2\eta} = \frac{1}{2}\eta \int d^3_+\tau \Big(\delta'(T) + iy_\alpha z^\alpha \delta(T)\Big) \exp(X^{loc}) C(Y_1; K) \bar{*} C(Y_2; K)\Big|_{Y_{1,2}=0},$$

$\bar{*}$ - "anti-holomorphic" star-product, $T = 1 - \sum_{i=1}^3 \tau_i$

$$X^{loc} = i\tau_3 z_\alpha y^\alpha + \tau_3 z^\alpha(\partial_{1\alpha} + \partial_{2\alpha}) + y^\alpha(\tau_2\partial_{2\alpha} - \tau_1\partial_{1\alpha}) + i\tau_3\partial_{1\alpha}\partial_2^\alpha\,.$$

Dynamical equations are Z-independent $\quad\Rightarrow\quad$ integral over τ_3 is finally located at $\tau_3 = 0$. $\quad\Rightarrow$ all terms $\sim \tau_3$ must disappear

$\Rightarrow\quad$ Dependence on $\partial_{1\alpha}\partial_2^\alpha$ vanishes

Exponential Representation and Z-Dominance Lemma

To control locality it suffices to consider the exponential parts of the operators acting on $C(y,\bar{y})$ focusing on the derivatives p^j $(\bar{p}^j)$.

Exponential representation in the holomorphic sector $\bar{\eta}=0$

$$\underbrace{C(y)*\ldots*C(y)}_{n} = \exp i\Big(-\sum_j p^j{}_\alpha y^\alpha - \mathcal{P}^n(p)\Big)C(y_1)\ldots C(y_n)\Big|_{y_j=0}$$

$$p^j{}_\alpha := i\frac{\partial}{\partial y_j^\alpha}\,, \qquad \mathcal{P}^n(p) = \frac{1}{2}\sum_{j,k} p^j{}_\alpha p^{k\alpha}$$

Solving $\mathsf{d}_Z U_n(Z;Y|dZ) = V[U_{j<n}](Z;Y|dZ)$ we use

Z**-dominance Lemma:** All terms dominated by the coefficients in front of the Z-dependent terms in the exponential factors in $U_n(Z;Y;K|x)$ **trivialize** in the field equations on the dynamical fields valued in the d_z-cohomology.

Parity of Exponential Parts

Corrections to dynamical equations to any perturbation order are constructed inductively, starting from

$$\gamma = \exp\left(iz_\alpha y^\alpha\right) k\theta^2 \quad \& \quad C(y,\bar{y}|x) \quad \& \quad W_0(y,\bar{y}|x) \quad \& \quad \omega(y,\bar{y}|x)$$

applying star-product, shift resolutions and d_x-differentiation.

Exponential parts of resulting expressions in the holomorphic sector can be classified as:

Odd class $\mathcal{E}^1_n$: $\exp i(T(\tau)z_\gamma y^\gamma + A_j(\tau)p^j_\gamma z^\gamma + B_j(\tau)p^j_\gamma y^\gamma + \frac{1}{2}\mathcal{P}_{ij}(\tau)p^i_\gamma p^{j\gamma})k^{n+1}$

$$\sum_{j=1}^{n}(-1)^j A_j = 0\,, \qquad \sum_{j=1}^{n}(-1)^j B_j = 1 - T\,, \qquad \sum_{i=1}^{n}(-1)^i \mathcal{P}_{ij} = -A_j$$

Even class $\mathcal{E}^0_n$: $\exp i(T(\tau)z_\gamma y^\gamma + A_j(\tau)p^j_\gamma z^\gamma + B_j(\tau)p^j_\gamma y^\gamma + \frac{1}{2}\mathcal{P}_{ij}(\tau)p^i_\gamma p^{j\gamma})k^{n}$

$$\sum_{j=1}^{n}(-1)^j A_j = -T\,, \qquad \sum_{j=1}^{n}(-1)^j B_j = 0\,, \qquad \sum_{i=1}^{n}(-1)^i \mathcal{P}_{ij} = B_j.$$

$$T \in \mathbb{C}\,, \qquad A, B \in \mathbb{C}^n\,, \qquad \mathcal{P}_{ij} = -\mathcal{P}_{ji} \in \mathbb{C}^n \times \mathbb{C}^n$$

Properties

$$(\bullet) \qquad \mathcal{E}^j_n * \mathcal{E}^i_m \subset \mathcal{E}^{(j+i)|_2}_{m+n}$$

Shifted resolution with $Q = -iv^j p^j + \mu y$

$$\Delta_Q(\ldots)E_n(T,A,B,\mathcal{P}) = \int_0^1 d\tau(\ldots)E_n(\tau T, \tau A, B-(1-\tau)Tv-(1-\tau)\mu A, \mathcal{P}+\widetilde{\mathcal{P}}),$$

$$\widetilde{\mathcal{P}}_{ij} = (1-\tau)\left(A_j v_i - A_i v_j\right)$$

$$E_n(T,A,B,\mathcal{P},p|z,y) = \exp i(Tz_\gamma y^\gamma + A_j p^j_\gamma z^\gamma + B_j p^j_\gamma y^\gamma + \frac{1}{2}\mathcal{P}_{ij}p^i_\gamma p^{j\gamma})$$

generates mapping $\mathcal{M}_{Q,\tau} : E_n(T,A,B,\mathcal{P})k^m \to E_n(T',A',B',\mathcal{P}')k^m$

$$(\bullet\bullet) \qquad \sum_{j=1}^n (-1)^j v^1_j = -1 \quad \Rightarrow \quad \mathcal{M}_{Q,\tau} : \mathcal{E}^1_n \to \mathcal{E}^1_n \qquad \forall \tau, \mu$$

$$(\bullet\bullet\bullet) \qquad \sum_{j=1}^n (-1)^j v^0_j = \mu \quad \Rightarrow \quad \mathcal{M}_{Q,\tau} : \mathcal{E}^0_n \to \mathcal{E}^0_n \qquad \forall \tau, \mu$$

$\mu = -1$ **: the two conditions are equivalent**

Analogously one can define mappings generated by

d_x: $\mathsf{d}_x C(Y_1)\ldots C(Y_n) = \sum_i C(Y_1)\ldots \widehat{C(Y_i)}\ldots C(Y_n)\Big|_{C(Y_i)\to\sum_j J^1_m(Y_i)}$

by virtue of field equations $\mathsf{d}_x C(Y;K|x) = \sum_{m=1}^n J^1_m(Y;K|x)$

defines mapping

- $S_{i,E^1_m(0,0,B,P,p|0,y)} : \mathcal{E}^p_n \to \mathcal{E}^p_{n+m-1}\,, \qquad p = 0,1$

Structure Lemma

If the perturbative analysis in the one-form sector contains shifted resolutions $\dot{\Delta}$ preserving even class, while that in the zero-form sector contains shifted resolutions preserving odd class, then all B_j generate odd exponentials, while all space-time zero- and one-form components of $\mathcal{W}_j$, generate even exponentials in the holomorphic sector.

Pfaffian Locality in the Holomorphic Sector

By definition odd fields contain $\mathcal{P}_{ij}$ satisfying $\sum_{i=1}^{n}(-1)^i\mathcal{P}_{ij} = -A_j$.

Coefficients A_j which are in front of the $z^\alpha\partial^j{}_\alpha$

$\Rightarrow$ trivialize in the field equations

$\Rightarrow$ $\mathcal{P}_{ij}$ is degenerate having the smaller degree of nonlocality than $C(y_1) * \ldots * C(y_n)$ at least for even n.

In the second order Pfaffian locality implies usual locality.

To obtain a local form of dynamical equations in the zero-form sector to the second order it is necessary to use shifted resolution

$$\Delta_{\beta y+i\alpha\partial_{y_1}-i(1-\alpha)\partial_{y_2}} \qquad \forall\beta\,,\ \alpha.$$

Details in the talks of Didenko and Koributh

Conclusion

The class of shifted homotopy operators is introduced

Z-Dominance Lemma: all terms dominated by those of the Z-dependent terms disappear in the dynamical field equations

Pfaffian Locality Theorem describing a class of shifted resolutions decreasing the level of nonlocality is proven

Structure Lemma is proven stating that at certain conditions on shifted homotopies all B_j generate odd exponentials, while all space-time zero- and one-form components of $\mathcal{W}_j$, generate even exponentials in the holomorphic sector.