

Towards Scattering Amplitudes from the Multiparticle Higher-Spin Charges

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5 June, 2018

Introduction

0. Why amplitudes and HS multiparticle charges may be related?
1. MHV amplitudes
2. Unfolded higher-spin equations
 - ▶ rank-1 fields \mathcal{C}
 - ▶ rank- r (multiparticle) fields $\mathcal{C}^{\otimes r}$
- 3 On-shell-closed differential forms Ω : $d\Omega|_{\text{unfold. eq-ns}} = 0$.
Conserved charges $\mathcal{Q} = \int_{\Sigma} \Omega$
- 4 Example of reproducing MHV amplitudes

Why \mathcal{A} 's and \mathcal{Q} 's may be related?

Scattering amplitudes	Multiparticle HS charges
Elements of the gauge-invariant S-matrix parametrized by multiparticle asymptotic (free) states	Gauge-invariant conserved quantities built of tensor products of free higher-spin fields
Constructed from classical on-shell data of free fields: light-like momenta (for external legs) and propagators	Constructed with the aid of classical dynamical equations for free higher-spin fields
Matrix elements of the operator $\exp(i \int d^4x L)$, rational functions of p	Objects $\int \Omega$ constructed from fields C encoding infinitely many derivatives

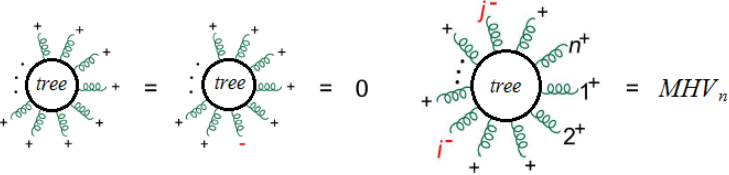
Paradox?

Does a single free classical HS theory enables one to calculate scattering amplitudes of various QFT's?

NO: Free multiparticle HS theory with infinitely many conserved charges is expected to give large enough “kinematical” space, able to contain (reproduce) any scattering amplitude of a given QFT.

MHV amplitudes

- ▶ Tree-level color-ordered n -particle amplitudes at the leading order in N in the $SU(N)$ Yang-Mills theory
- ▶ One-particle on-shell data is given by light-like momentum $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$, polarization $\epsilon_{a\dot{a}}^+ \sim \chi_a \tilde{\lambda}_{\dot{a}}$ or $\epsilon_{a\dot{a}}^- \sim \lambda_a \tilde{\chi}_{\dot{a}}$ (in terms of $SL(2, \mathbb{C})$ spinors) and color.
- ▶ For the specific helicity configurations (outgoing particles)



MHV amplitudes

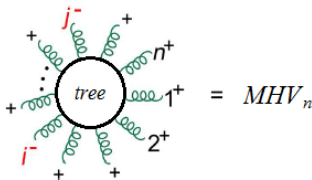
- ▶ Expression for MHV_n is given by Parke-Taylor formula (Parke and Taylor '86, proven by Berends, Giele '88)

$$\mathcal{A} \left(1_{p_1, c_1}^+ \cdots c_{p_i, c_i}^- \cdots c_{p_j, c_j}^- \cdots c_{p_n, c_n}^+ \right) \propto \text{Tr} (T^{c_1} \cdots T^{c_n}) \mathcal{A}_{\lambda, \tilde{\lambda}}^{(ij)}$$

where

$$\mathcal{A}_{\lambda, \tilde{\lambda}}^{(ij)} = \frac{\langle \lambda_i \lambda_j \rangle^4}{\langle \lambda_1 \lambda_2 \rangle \langle \lambda_2 \lambda_3 \rangle \cdots \langle \lambda_n \lambda_1 \rangle} \delta^{(4)} \left(\sum_{l=1}^n \lambda_l \tilde{\lambda}_l \right),$$

with $\langle \lambda_r \lambda_s \rangle = \varepsilon^{ab} \lambda_{ra} \lambda_{sb}$ and $p_{l a \dot{a}} = \lambda_{l a} \tilde{\lambda}_{l \dot{a}}$



MHV amplitudes

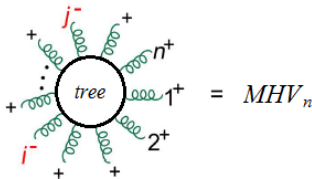
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with $\langle \lambda_r \lambda_s \rangle = \varepsilon^{ab} \lambda_{ra} \lambda_{sb}$ and $p_{l a \dot{a}} = \lambda_{la} \tilde{\lambda}_{l \dot{a}}$



Unfolded HS equations (rank 1)

- ▶ rank-1 fields: gauge invariant 0-forms obeying

$$\left(\frac{\partial}{\partial x^{a\dot{a}}} + i \frac{\partial^2}{\partial y^a \partial \tilde{y}^{\dot{a}}} \right) C^1(y, \tilde{y}|x) = 0,$$

where $a, \dot{a} = 1, 2$, $x^{a\dot{a}}$ are space-time coordinates, $y^a, \tilde{y}^{\dot{a}}$ are auxiliary variables.

N.B. Usually formal power series in y, \tilde{y}

$$C^1(y, \tilde{y}|x) = \sum_{m,n} \frac{c_{a(m), \dot{a}(n)}(x)}{m!n!} y^a \dots y^a \tilde{y}^{\dot{a}} \dots \tilde{y}^{\dot{a}},$$

unfolded equations encode the set of primary fields $c_{a(n)}(x)$ and $c_{\dot{a}(n)}(x)$ (anti-self-dual and self-dual HS curvatures) and dynamical equations on primaries

$$i \frac{\partial}{\partial x^{a\dot{a}}} \sim \mu_a \tilde{\mu}_{\dot{a}}$$

Unfolded HS equations (rank 1)

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N.B. In what follows $C^1(y, \tilde{y}|x)$ is a complex-valued distribution, variables y, \tilde{y} are involved into play. Plane-wave solutions

$$\exp \left[i \left(\mu_a \tilde{\mu}_{\dot{a}} x^{a\dot{a}} + \mu_a y^a + \tilde{\mu}_{\dot{a}} \tilde{y}^{\dot{a}} \right) \right]$$

obey light-like-momentum constraint

$$i \frac{\partial}{\partial x^{a\dot{a}}} \sim \mu_a \tilde{\mu}_{\dot{a}}$$

Unfolded HS equations (rank \mathbf{r})

- rank- \mathbf{r} fields: tensor products of \mathbf{r} rank-1 fields obeying

$$\left(\frac{\partial}{\partial x^{a\dot{a}}} + i \delta_{IJ} \frac{\partial^2}{\partial y_I^a \partial \tilde{y}_J^{\dot{a}}} \right) C^{\mathbf{r}}(y, \tilde{y}|x) = 0,$$

where $I, J = 1 \dots \mathbf{r}$. Particular solutions are products

$$C^{\mathbf{r}}(y, \tilde{y}|x) = C^1(y_1, \tilde{y}_1|x) \dots C^1(y_{\mathbf{r}}, \tilde{y}_{\mathbf{r}}|x),$$

for example plane-wave solutions

$$\chi_{\mu, \tilde{\mu}} = \exp \left[i \sum_{l=1}^{\mathbf{r}} (\mu_{l\dot{a}} \tilde{\mu}_{l\dot{a}} x^{a\dot{a}} + \mu_{l\dot{a}} y^{l\dot{a}} + \tilde{\mu}_{l\dot{a}} \tilde{y}^{l\dot{a}}) \right]$$

Unfolded HS equations (rank \mathbf{r})

- ▶ For a partition $\mathbf{r} = m + \bar{m}$ consider half-Fourier transform

$$y_{1\dots m}, \underbrace{y_{m+1\dots r}}_{\bar{m}} \quad \text{and} \quad \tilde{y}_{1\dots \bar{m}}, \underbrace{\tilde{y}_{\bar{m}+1\dots r}}_m$$

to

$$\lambda_{1\dots m}, y_{1\dots \bar{m}} \quad \text{and} \quad \tilde{\lambda}_{1\dots \bar{m}}, \tilde{y}_{\bar{m}+1\dots r}$$

- ▶ Rank- \mathbf{r} unfolded equation becomes a first-order PDE

$$\left(\frac{\partial}{\partial x^{a\dot{a}}} - \lambda_{i\dot{a}} \frac{\partial}{\partial \tilde{y}_i^{\dot{a}}} - \tilde{\lambda}_{\bar{i}\dot{a}} \frac{\partial}{\partial y_{\bar{i}}^{\dot{a}}} \right) g^{(m,\bar{m})} \left(\lambda_i, y_{\bar{i}}, \tilde{y}_{\bar{j}}, \tilde{\lambda}_{\bar{j}} \mid x \right) = 0$$

with $i, j = 1 \dots m$ and $\bar{i}, \bar{j} = 1 \dots \bar{m}$, with characteristics

$$\lambda_{i\dot{a}}, \quad \tilde{\lambda}_{\bar{i}\dot{a}}, \quad y_{\bar{i}}^{\dot{a}} + x^{a\dot{a}} \tilde{\lambda}_{\bar{i}\dot{a}}, \quad \tilde{y}_{\bar{i}}^{\dot{a}} + x^{a\dot{a}} \lambda_{i\dot{a}}$$

Unfolded HS equations (rank r)

- ▶ Rank- r unfolded equation as a first-order PDE

$$\left(\frac{\partial}{\partial x^{a\dot{a}}} - \lambda_{i a} \frac{\partial}{\partial \tilde{y}_i^{\dot{a}}} - \tilde{\lambda}_{\bar{i} \dot{a}} \frac{\partial}{\partial y_{\bar{i}}^a} \right) g^{(m, \bar{m})} (\lambda_i, y_{\bar{i}}, \tilde{y}_j, \tilde{\lambda}_{\bar{j}} | x) = 0$$

with $i, j = 1 \dots m$ and $\bar{i}, \bar{j} = 1 \dots \bar{m}$. Characteristics

$$\lambda_{i a}, \quad \tilde{\lambda}_{\bar{i} \dot{a}}, \quad y_{\bar{i}}^a + x^{a\dot{a}} \tilde{\lambda}_{\bar{i} \dot{a}}, \quad \tilde{y}_i^{\dot{a}} + x^{a\dot{a}} \lambda_{i a}$$

- ▶ Plane-wave solution

$$\chi_{\mu, \tilde{\mu}} = \exp \left[i \left(\tilde{\mu}_{\dot{a}}^i (\tilde{y}_i^{\dot{a}} + x^{a\dot{a}} \lambda_{i a}) + \mu_a^{\bar{j}} (y_{\bar{j}}^a + x^{a\dot{a}} \tilde{\lambda}_{\bar{j} \dot{a}}) \right) \right] \cdot \prod_{j, a} \delta (\lambda_a^j - \mu_a^j) \prod_{\bar{j}, \dot{a}} \delta (\tilde{\lambda}_{\dot{a}}^{\bar{j}} - \tilde{\mu}_{\dot{a}}^{\bar{j}})$$

- ▶ Useful x -independent solution $\rho_{i\bar{j}} = \lambda_{i a} y_{\bar{j}}^a - \tilde{\lambda}_{\bar{j} \dot{a}} \tilde{y}_i^{\dot{a}}$
(Arkani-Hamed et al.)

Conserved charges

- ▶ Turning unfolded equations into the system of first-order PDE's gives a number of useful options:

1. On-shell-closed differential form

$$\Omega^{(m,\bar{m})} [g] = d^{K^m} \lambda d^{K^{\bar{m}}} \tilde{\lambda} d^{K^{\bar{m}}} (y + x\tilde{\lambda}) d^{K^m} (\tilde{y} + x\lambda) g^{(m,\bar{m})}$$

2. For a given solution $g^{(m,\bar{m})}$ there is functional ambiguity $g^{(m,\bar{m})} \rightarrow \eta g^{(m,\bar{m})}$, where “parameter” η is a solution of rank- r equation.

- ▶ Integration

$$Q^{(m,\bar{m})} = \int_{\Sigma} \Omega^{(m,\bar{m})}$$

is independent of local variations of Σ , giving multiparticle conserved charge.

Reproducing MHV 3-point amplitude

$$Q^{(m,\bar{m})} = \int_{\Sigma} d^{Km} \lambda d^{K\bar{m}} \tilde{\lambda} d^{K\bar{m}} (y + x\tilde{\lambda}) d^{Km} (\tilde{y} + x\lambda) g^{(m,\bar{m})}$$

- ▶ $r = 3$, consider $m = 2, \bar{m} = 1$: $j = 1, 2, \bar{j} = 3$
- ▶ Σ such that $x = 0$
- ▶ $g^{(2,1)} = \text{sign}(\rho_{13}) \text{sign}(\rho_{23}) \chi_{\mu, \tilde{\mu}}$

$$Q^{(2,1)} = \int d^2 y d^4 \tilde{y} \exp(i\tilde{\mu}_1 \tilde{y}_1 + i\tilde{\mu}_2 \tilde{y}_2 + i\mu_3 y_3) \\ \text{sign}(\mu_1 y_3 - \tilde{\mu}_3 \tilde{y}_1) \text{sign}(\mu_2 y_3 - \tilde{\mu}_3 \tilde{y}_2).$$

- ▶ Change of 6 variables from y_i^a, \tilde{y}_j^a to 2 variables ρ_{13}, ρ_{23} and 4 variables $\tilde{\zeta}_i^{\dot{a}}$ parametrizing the kernel of

$$\rho_{i\bar{j}} = \mu_{i a} y_j^a - \tilde{\mu}_{\bar{j} \dot{a}} \tilde{y}_i^{\dot{a}}$$

Reproducing MHV 3-point amplitude

$$Q^{(2,1)} = \int d^2y d^4\tilde{y} \exp(i\tilde{\mu}_1\tilde{y}_1 + i\tilde{\mu}_2\tilde{y}_2 + i\mu_3y_3) \\ \text{sign}(\mu_1y_3 - \tilde{\mu}_3\tilde{y}_1) \text{sign}(\mu_2y_3 - \tilde{\mu}_3\tilde{y}_2).$$

- ▶ Change of variables gives the following Jacobian

$$d^2y d^4\tilde{y} = \langle\mu_1\mu_2\rangle^5 d^2\rho d^4\tilde{\xi}$$

- ▶ Exponent transforms into

$$\exp\left[i\rho_{\bar{i}\bar{j}}\langle\mu_{\bar{j}}\varepsilon_{ik}\mu_k\rangle\right] \exp\left[i\tilde{\xi}_{\bar{i}}^{\bar{a}}F_{i\bar{a}}\right],$$

where $F_{i\bar{a}} = \langle\mu_1\mu_2\rangle\tilde{\mu}_{i\bar{a}} + \langle\mu_{\bar{j}}\varepsilon_{ik}\mu_k\rangle\tilde{\mu}_{\bar{j}\bar{a}}$

- ▶ We make use of the integrals

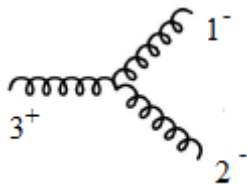
$$\int_{-\infty}^{+\infty} dx e^{ikx} \text{sign}(x) = \frac{2i}{k} \quad \text{and} \quad \int_{-\infty}^{+\infty} dx e^{ikx} = 2\pi \delta(k)$$

Reproducing MHV 3-point amplitude

We arrive at the desired result

$$Q^{(2,1)} \propto \frac{\langle \mu_1 \mu_2 \rangle^3}{\langle \mu_2 \mu_3 \rangle \langle \mu_3 \mu_1 \rangle} \delta^4 \left(\sum_I \mu_{Ia} \tilde{\mu}_{I\dot{a}} \right),$$

of the MHV 3-point function



Reproducing MHV r -point amplitude

- ▶ $m = 2, \bar{m} = r - 2$
- ▶ Σ such that $x = 0$
- ▶ $g^{(2,r-2)} = \prod_{i,\bar{j}} \text{sign}(\rho_{i\bar{j}}) \chi_{\mu,\tilde{\mu}}$
- ▶ ...

However

$$Q^{(2,r-2)} \propto \frac{\langle \mu_1 \mu_2 \rangle^r}{\prod_{i=1,2,\bar{j}=\bar{3}\dots\bar{n}} \langle \mu_i \mu_{\bar{j}} \rangle} \delta^4 \left(\sum_I \mu_{Ia} \tilde{\mu}_{I\dot{a}} \right)$$

- ▶ We still have ambiguity in the choice of “parameter” η :
rational function

$$\eta_{\mu}^{\text{extra}} = \frac{\langle \lambda_1 \mu_{\bar{3}} \rangle \dots \langle \lambda_1 \mu_{\bar{r}} \rangle \cdot \langle \lambda_2 \mu_{\bar{3}} \rangle \dots \langle \lambda_2 \mu_{\bar{r}} \rangle}{\langle \lambda_1 \lambda_2 \rangle^{r-3} \langle \lambda_2 \mu_{\bar{3}} \rangle \langle \mu_{\bar{4}} \mu_{\bar{5}} \rangle \dots \langle \mu_{\bar{r}-1} \mu_{\bar{r}} \rangle \langle \mu_{\bar{r}} \lambda_1 \rangle}$$

brings $Q^{(2,r-2)}$ to MHV r -point amplitude.

Conclusion

- ▶ Scattering amplitudes and multiparticle HS charges are indeed related: examples of reproducing MHV amplitudes
- ▶ There is a vast ambiguity in the construction of Ω , giving possibility to reproduce amplitudes by adjusting “parameter” η
- ▶ Spin information has to be specified within the formalism
- ▶ Particular form of amplitudes may be fixed only by considering non-linear equations