PERTURBATIVE SCHEMES, ORDERINGS AND GAUGE FUNCTIONS IN HS GRAVITY

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Motivations

- Presentend a wide solution space that can be thought of as resulting from a resummation of a perturbative expansion which differs in various respects from the usual one.

- Understanding the connection between these two perturbative approaches may clarify the physical interpretation of those solutions (as well as related crucial formal issues, such as determining an admissible class of gauge functions, etc. ...).

- As a case study, we shall focus on massless particle solutions of the linearized theory and use them to understand some feature of the alternative perturbation scheme.

Kinematics

- Master-fields living on *correspondence space*, locally $\mathcal{X}$ x $\mathcal{Z}$ x $\mathcal{Y}$:

$$\widehat{W} = dx^\mu \widehat{W}_\mu(Y,Z|x)$$ → gauge fields of all spins + auxiliary

$$\widehat{\Phi} = \widehat{\Phi}(Y,Z|x)$$ → Weyl tensors and their derivatives → local dof

$$\widehat{S} = dz^\alpha \widehat{S}_\alpha(Y,Z|x) + d\bar{z}^{\dot\alpha} \widehat{\bar{S}}_{\dot\alpha}(Y,Z|x)$$ → Z-space connection, no extra local dof

- Commuting oscillators $Y_{\underline{\alpha}} = (y_\alpha, \bar{y}_{\dot\alpha})\ ,\quad Z_{\underline{\alpha}} = (z_\alpha, -\bar{z}_{\dot\alpha})$ → $\mathfrak{sp}$(4,R) quartets

$$[Y_{\underline{\alpha}}, Y_{\underline{\beta}}]_\star = 2iC_{\underline{\alpha\beta}} = 2i \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon_{\dot\alpha\dot\beta} \end{pmatrix}\ , \qquad [Z_{\underline{\alpha}}, Z_{\underline{\beta}}]_\star = -2iC_{\underline{\alpha\beta}}\ , \quad [Y_{\underline{\alpha}}, Z_{\underline{\beta}}]_\star = 0$$

- Star-product:

$$\widehat{F}(Y,Z) \star \widehat{G}(Y,Z) = \int_{\mathcal{R}} \frac{d^4U d^4V}{(2\pi)^4}\, e^{iV^{\underline{\alpha}}U_{\underline{\alpha}}}\, \widehat{F}(Y+U, Z+U)\, \widehat{G}(Y+V, Z-V)$$

- Inner kleinian operator $\hat{\kappa}$:

$$\widehat{\kappa} = e^{iy^\alpha z_\alpha}\ , \qquad \widehat{\kappa} \star \widehat{f}(z,y) = \widehat{f}(-z,-y) \star \widehat{\kappa}\ , \qquad \widehat{\kappa} \star \widehat{\kappa} = 1$$

$$\widehat{\kappa} = \kappa_y \star \kappa_z\ , \quad \kappa_y \star \widehat{f}(z,y) = \widehat{f}(z,-y) \star \kappa_y\ , \quad \kappa_y \star \kappa_y = 1\ ,$$

$$\kappa_y = 2\pi\delta^2(y) = 2\pi\delta(y_1)\delta(y_2)$$

4D bosonic Vasiliev equations

- Full equations:

$$\boxed{\begin{aligned} d\widehat{W}+\widehat{W}\star\widehat{W} &= 0\\ d\widehat{\Phi}+\widehat{W}\star\widehat{\Phi}-\widehat{\Phi}\star\pi(\widehat{W}) &= 0\\ d\widehat{S}_\alpha+[\widehat{W},\widehat{S}_\alpha]_\star &= 0\\ \widehat{S}_\alpha\star\widehat{\Phi}+\widehat{\Phi}\star\pi(\widehat{S}_\alpha) &= 0\\ [\widehat{S}_\alpha,\widehat{S}_\beta]_\star &= -2i\epsilon_{\alpha\beta}(1-b\widehat{\Phi}\star\widehat{\kappa})\\ [\widehat{S}_\alpha,\widehat{\bar{S}}_{\dot\beta}]_\star &= 0\ , \end{aligned}}$$

(Vasiliev, '92)

$$\boxed{\widehat{S}_\alpha \;=\; z_\alpha-2i\widehat{V}_\alpha}$$

$$[\widehat{S}_\alpha,\widehat{f}(Z,Y)]=[z_\alpha,\widehat{f}]+\ldots=-2i\frac{\partial}{\partial z^\alpha}\widehat{f}+\ldots$$

- *Z*-oscillators → auxiliary, non-commutative coordinates. Equations fix the evolution along Z in such a way that it gives rise to consistent interactions to all orders among physical fields, contained in the (Z-independent) initial conditions

$$W \;=\; \widehat{W}_{|Z=0}\ , \qquad \Phi \;=\; \widehat{\Phi}_{|Z=0}\ .$$

- 1st order eqs impose a relation between spacetime and twistor space behaviour of their solutions → the physical information can be encoded to a great extent in the twistor-space dependence.

AdS vacuum solution

$$\boxed{\begin{aligned} \widehat{\Phi} &= \widehat{\Phi}^{(0)} = 0\,, \\ \widehat{S}_{\alpha} = \widehat{S}^{(0)}_{\alpha} = z_{\alpha}\,, &\qquad \widehat{S}_{\dot\alpha} = \widehat{S}^{(0)}_{\dot\alpha} = \bar z_{\dot\alpha}\,, \\ \widehat{W}_{\mu} = \Omega^{(0)}_{\mu} &= \tfrac{1}{4i}\left(\omega^{(0)\alpha\beta}_{\mu} y_{\alpha} y_{\beta} + \bar\omega^{(0)\dot\alpha\dot\beta}_{\mu}\bar y_{\dot\alpha}\bar y_{\dot\beta} + 2e^{(0)\alpha\dot\beta}_{\mu} y_{\alpha}\bar y_{\dot\beta}\right) \end{aligned}}$$

$$e^{\alpha\dot\beta}_{(0)} = -\frac{dx^{a}(\sigma_{a})^{\alpha\dot\beta}}{1-x^{2}}\,, \qquad \omega^{\alpha\beta}_{(0)} = \frac{x^{a}dx^{b}(\sigma_{ab})^{\alpha\beta}}{1-x^{2}}$$

$$\longrightarrow \quad \boxed{ds^{2}_{(0)} = \frac{4dx^{2}}{(1-x^{2})^{2}}}$$

- $\widehat{W}$ is a flat connection, can be represented via a gauge function *L*(x|Y) = *AdS*$_4$ coset element

$$\boxed{\widehat{W}_{\mu} = \Omega^{(0)}_{\mu} = L^{-1}\star\partial_{\mu}L}$$

$$L(x;y,\bar y) = e_{\star}^{i\tilde x^{\mu}(x)\delta^{a}_{\mu}P_{a}} \quad : \quad \mathcal{R}^{3,1} \longrightarrow \frac{SO(3,2)}{SO(3,1)}$$

In Weyl ordering: $$L(x;y,\bar y) = \frac{2h}{1+h}\exp\left(\frac{x^{\alpha\dot\beta}}{1+h}y_{\alpha}\bar y_{\dot\beta}\right)\,, \qquad h := \sqrt{1-x^{2}}$$

Perturbative analysis

- At first order

$$\begin{aligned} D^{(0)}\widehat{W}^{(1)} &= 0 \\ D^{(0)}\widehat{\Phi}^{(1)} &= 0 \\ d_Z\widehat{W}^{(1)} &= -D^{(0)}\widehat{V}^{(1)} \\ d_Z\widehat{\Phi}^{(1)} &= 0 \\ d_Z\widehat{V}^{(1)} &= -\frac{ib}{4}\widehat{\Phi}^{(1)}\star\widehat{J}\,, \qquad \widehat{J} \;:=\; -\frac{i}{4}dz^2\kappa - \text{h.c.} \end{aligned}$$

- The eqs. with at least one component on Z can be integrated to give the hatted fields in terms of non-linear couplings involving the original dof in Φ:

$$\begin{aligned} \widehat{\Phi}^{(1)} &= \Phi(x,Y)\,, \\ \widehat{V}^{(1)} &= d_Z\xi + dz^\alpha z_\alpha\int_0^1 dt\,t\,\Phi(-tz,\bar{y})\,e^{ity^\alpha z_\alpha} + \text{h.c.} \;=:\; d_Z\xi + \rho(\Phi\star\widehat{J})\,, \quad \rho := i_Z\frac{1}{\mathcal{L}_Z} \\ \widehat{W}^{(1)} &= \omega(Y) - \rho D^{(0)}\widehat{V}^{(1)} \;=\; \omega(Y) + D^{(0)}\xi - \underbrace{\rho D^{(0)}\rho(\Phi\star\widehat{J})}_{\xrightarrow[Z\to 0]{}\;0}\,, \end{aligned}$$

- The gauge ambiguity on V translates into a gauge ambiguity on the gauge field generating function

$$\widehat{W}^{(1)}\Big|_{Z=0} \;=\; \omega(Y) + D^{(0)}\xi\Big|_{Z=0}$$

which can be fixed by requiring ξ = ξ (Y) → *Vasiliev gauge,* $\boxed{i_Z\widehat{V}^{(1)} \;\equiv\; Z^{\underline{\alpha}}\widehat{V}^{(1)}_{\underline{\alpha}} \;=\; 0}$

Perturbative analysis

- Substitution in the remaining equations gives Klein-Gordon, Maxwell, linearized Einstein and Fronsdal eqs. in unfolded form → *Central On-Mass-Shell Theorem* (COMST) :

$$\boxed{\begin{aligned} d\omega + \{\Omega^{(0)}, \omega\}_\star &= -\frac{i\bar{b}}{4} H^{\alpha\beta} \frac{\partial^2}{\partial y^\alpha \partial y^\beta} \Phi|_{\bar{y}=0} - \frac{ib}{4} \bar{H}^{\dot{\alpha}\dot{\beta}} \frac{\partial^2}{\partial \bar{y}^{\dot{\alpha}} \partial \bar{y}^{\dot{\beta}}} \Phi|_{y=0} \\ d\Phi + \Omega^{(0)} \star \Phi - \Phi \star \pi(\Omega^{(0)}) &= 0 \end{aligned}}$$

$$H^{\alpha\beta} := e^{(0)\alpha\dot{\gamma}} e^{(0)\beta}{}_{\dot{\gamma}}$$

- The twisted adjoint equation already contains the information on the free propagation of all spin-s fields via the Bargmann-Wigner eqs. on the curvatures. The first equation is a gluing of the Weyl module to the gauge-field module (via the Chevalley-Eilenberg cocycle).

- On the other hand, all exact solutions of the theory have been obtained in different gauges than the Vasiliev gauge, and following the opposite route: working in the full (x,Y,Z)-space in order to take advantage of the simplicity of the eqs. and of the huge gauge freedom of the theory.

Exact solutions: gauge function method

- Takes maximum advantage from the fact that the physics is to a large extent encoded in twistor space.
- $\mathcal{X}$ x $\mathcal{Y}$ x $\mathcal{Z}$-space eqns:

$$\widehat{W} = \widehat{g}^{-1} \star d\widehat{g}$$

$$\widehat{\Phi} = \widehat{g}^{-1} \star \widehat{\Phi}' \star \pi(\widehat{g}) \,, \qquad d\widehat{\Phi}' = 0$$

$$\widehat{S}_\alpha = \widehat{g}^{-1} \star \widehat{S}'_\alpha \star \widehat{g} \,, \qquad d\widehat{S}'_\alpha = 0$$

- $\mathcal{Y}$ x $\mathcal{Z}$-space eqns:

$$\widehat{S}'_\alpha \star \widehat{\Phi}' + \widehat{\Phi}' \star \pi(\widehat{S}'_\alpha) = 0$$

$$[\widehat{S}'_\alpha, \widehat{S}'_\beta]_\star = -2i\epsilon_{\alpha\beta}(1 - b\,\widehat{\Phi}' \star \widehat{\kappa})$$

$$[\widehat{S}'_\alpha, \widehat{\bar{S}}'_{\dot{\beta}}]_\star = 0$$

(Vasiliev, Sezgin-Sundell, C.I.-Sundell, Giombi-Yin…)

- Solve locally all equations with at least one spacetime component via some gauge function $\hat{g} = \hat{g}(x,Y,Z)$.
- Then solve the Z-space constraints to determine the spacetime constants $\hat{\Phi}' = \hat{\Phi}'(Y,Z)$ and $\hat{S}'_\alpha = \hat{S}'_\alpha(Y,Z)$.
- Working in terms of primed fields → *W=0 gauge*.

  Easier to build solutions in W=0 gauge, the equations are algebraic. In order to read any spacetime feature (correlation functions, asymptotic charges,…) change gauge and reinstate x-dep. by performing the star-products with $\hat{g}$.

Exact solutions: gauge function method

- The behaviour of the fields in (x,Y,Z) depends on a subtle interplay of gauge function and primed fields.

- At fixed gauge function, it is the twistor-space behaviour of the initial Φ' and S'_α that determines the spacetime behaviour of the fields.
  Indeed, certain observables (such as some that can be used to endow the space of solutions with a norm) only depend on primed fields.

- Other important aspects of the interpretation however crucially depend on the choice of the gauge function. Interpreting the theory as a deformed Fronsdal theory seems to select a preferred gauge function implementing the Vasiliev gauge. What is the gauge function corresponding to such choice?

  → Restrictions on physically admissible solutions as well as determining the superselection sectors of the theory coincides with selecting classes of functions in twistor space. It is to be understood which are the requirements that separate admissible vs. inadmissible gauge transf.

Factorized expansion in holomorphic gauge

- A large solution space of interesting solutions (including HS black holes, HSbh + massless scalar, some cosmology-like solutions,…) take the form :

$$\begin{aligned}
\widehat{\Phi}'(Y,Z) &= \Phi'(Y) \, , \\
\widehat{V}'_\alpha(Y,Z) &= \widehat{V}'_\alpha(Y,z) = \widehat{V}'_\alpha(\Psi(Y),z) = \sum_{k=1}^{\infty} (\Psi(Y))^{\star k} \star V^{(k)}_\alpha(z) \, , \\
\widehat{\bar{V}'}_{\dot\alpha}(Y,Z) &= \widehat{\bar{V}'}_{\dot\alpha}(Y,\bar z) = \widehat{\bar{V}'}_{\dot\alpha}(\bar\Psi(Y),\bar z) = \sum_{k=1}^{\infty} (\bar\Psi(Y))^{\star k} \star \bar{V}^{(k)}_{\dot\alpha}(\bar z)
\end{aligned}$$

$$\Psi := \Phi' \star \kappa_y \, , \qquad \bar\Psi := \Phi' \star \bar\kappa_{\bar y} \, , \qquad [\Psi, \bar\Psi]_\star = 0$$

→ an all-order perturbative expansion in star-powers of the curvatures, absorbing all the *Y*-dependence, with separation of *Y* and *Z* variables and *V* holomorphic in *z*.

- Whereas the ordinary perturbative analysis is organized in powers of $\Phi\star\kappa$ and normal order, this can be considered an expansion in Ψ in Weyl order (no contractions between *Y* and *Z*).
- The different solutions are singled out by the different basis functions (or distributions) of *Y* variables on which one expands Φ' (i.e., Ψ) .
- The expansion in Ψ enables one to solve for the *Z* dependence in a universal way.

Factorized Ansatz in holomorphic gauge

- This is because Φ'=Φ'(Y) → the Z-dependence in the source term is universal and given by κ_z :

$$\partial_{[\alpha}\widehat{V}'_{\beta]}+\widehat{V}'_{[\alpha}\star\widehat{V}'_{\beta]} \;=\; -\frac{i}{4}\,\epsilon_{\alpha\beta}\,b\Psi\star\kappa_z$$

First order in Ψ:

$$\partial_{[\alpha}\widehat{V}^{(1)}_{\beta]} \;=\; -\frac{i}{4}\,\epsilon_{\alpha\beta}\,b\kappa_z$$

solved by a distributional z-space element

$$\boxed{\widehat{V}^{(1)\pm} \;\sim\; z^{\pm}\int_{-1}^{1}\frac{dt}{(t+1)^2}\,e^{i\frac{t-1}{t+1}z^+z^-} \;\sim\; \frac{1}{z^{\mp}}\lim_{\epsilon\to 0}(1-e^{-\frac{i}{\epsilon}z^+z^-}) \;\sim\; \theta(z^{\pm})\delta(z^{\mp})}$$

$$z^{\pm}:=u^{\pm\alpha}z_{\alpha}\;,\quad w_z:=z^+z^-\;,\quad [z^-,z^+]_{\star}=-2i \quad\rightarrow\quad \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}e^{-i\frac{1}{\varepsilon}z^+z^-} \;=\; \kappa_z$$

with basis spinors $u^{\pm}{}_{\alpha}$ ($u^{+\alpha}u^{-}{}_{\alpha}=1$) entering as a realization of a delta function in a Gaussian basis (one could have equally well used a plane wave basis, in which case an auxiliary spinor, the momentum associated to z, would have played that role).

- Higher orders:

$$\partial_{[\alpha}V^{(k)}_{\beta]}+\frac{1}{2}\sum_{p+q=k}\left[V^{(p)}_{\alpha},V^{(q)}_{\beta}\right]_{\star} \;=\; 0\;,\quad k\geq 2$$

$$\rightarrow\quad \sum_{k\geq 1}V^{(k)}_{\alpha}\star\Psi^{\star k} \;=\; \int_{-1}^{1}\frac{dt}{(t+1)^2}\,{}_1F_{1\star}(1/2;2;b\log t^2\Psi)\star z_{\alpha}\,e^{i\frac{t-1}{t+1}w_z}$$

Comments and observations

- The factorized expansion encodes a (formal) solution space in which Φ' is first-order exact, and the Z-dependence is solved in a universal way
  → gives a systematic procedure to non-linearly deform solutions of the KG and Bargmann-Wigner eqs. into solutions of the full Vasiliev eqs.

  This also facilitates their physical interpretation as well as the superposing of linearized twisted-adjoint sectors, e.g., $\Psi = \Psi_{bh} + \Psi_{part}$.

- Actual solutions must satisfy:
  1. The star-products $(F')^{*k}$ must be finite → conditions on the fiber algebra $\mathcal{A}(Y)$
  2. The zero-form charges should be finite (well-defined inner product)
  3. V_α should be at least real-analytic in Z.

  In the case that all Ψ^{*k} can be expanded over a common basis of functions, one can actually write down the full solution in closed form immediately.

- Further constraints placed by requiring the solution to correspond to an asymptotic configuration of Fronsdal fields (over AdS) → analiticity in Y and Z in Vasiliev gauge and finiteness of asymptotic charges.

Massless particle modes

- Factorized expansion already used to nonlinearly deform *massless scalar* modes + spherically symmetric *HS black holes*.

- Massless particle modes build up unitary $\mathfrak{so}(3,2)$ LW modules.

Unfolded Weyl 0-form equations, i.e., reformulation of the Bargmann-Wigner eqs. via a covariant constancy condition on the twisted adjoint module,

$$\Phi(x|Y) \;=\; L^{-1}(x) \star \Phi' \star \pi(L)(x)$$

show that particle modes can be encoded into specific algebraic elements: operators on singleton Fock space, non-polynomial functions of Y with definite eigenvalues under the Cartan subalgebra (E,J) of $\mathfrak{so}(3,2)$,

$$\Phi'(Y) \;\in\; \mathcal{M} \;=\; \bigoplus_{\mathbf{n},\mathbf{m}} \mathbf{C} \otimes P_{\mathbf{n}|\mathbf{m}}$$

$$P_{\mathbf{n}|\mathbf{n}'} \star P_{\mathbf{m}|\mathbf{m}'} \;=\; \delta_{\mathbf{n}',\mathbf{m}} P_{\mathbf{n}|\mathbf{m}'} \;, \qquad P_{\mathbf{n}|\mathbf{m}'} \;\sim\; |\mathbf{n}><\mathbf{m}| \;, \qquad \mathbf{n},\, \mathbf{m} \equiv (n_1, n_2), (m_1, m_2)$$

$$E \star P_{\mathbf{n}|\mathbf{m}} \;=\; \frac{n_1+n_2}{2} P_{\mathbf{n}|\mathbf{m}} \;, \qquad J \star P_{\mathbf{n}|\mathbf{m}} \;=\; \frac{n_2-n_1}{2} P_{\mathbf{n}|\mathbf{m}}$$

- Modules built by solving LW conditions $[L^{--}_r, P_{n|m}]_\pi = 0$ and then acting with L^{+}_r.
- This offers a simple way of solving for all the AdS-massless particle modes.

Massless scalar particle modes

- For example, the rotationally-invariant scalar field modes are encoded by projectors |n> <n|

$$\mathcal{P}_n(E) \;=\; 4(-)^{n-\frac{1+\varepsilon}{2}}\, e^{-4E} L^{(1)}_{n-1}(8E) \;=\; 2(-)^{n-\frac{1+\varepsilon}{2}} \oint_{C(\varepsilon)} \frac{d\eta}{2\pi i} \left(\frac{\eta+1}{\eta-1}\right)^n e^{-4\eta E}.$$

- Indeed, using the simple AdS gauge function *L* (*L-gauge*) we reconstruct exactly the Breitenlohner-Freedman scalar modes,

$$\Phi'(Y) \;=\; \Phi'_{s=0}(Y) \;=\; \sum_n \widetilde{\nu}_n\, \mathcal{P}_n(E)\;, \qquad (\widetilde{\nu}_n)^* \;=\; \widetilde{\nu}_{-n}$$

$$\Phi_{s=0}(x|Y) \;=\; L^{-1}(x) \star \Phi'_{s=0} \star \pi(L)(x) \;=\; (1-x^2) \sum_n \mathcal{N}_n \widetilde{\nu}_n \oint_{C(\epsilon)} \frac{d\eta}{2\pi i} \left(\frac{\eta+1}{\eta-1}\right)^n \frac{e^{iy^\alpha M_\alpha{}^{\dot\beta}(x,\eta)\bar y_{\dot\beta}}}{1-2i\eta x_0+\eta^2 x^2}$$

- For instance, the LW element n=1 ($\Phi' = 4e^{-4E}$) gives rise to the ground state of the D(1,0) scalar, as expected:

$$\boxed{4\widetilde{\nu}_1 \frac{1-x^2}{1-2ix_0+x^2} \;\sim\; \widetilde{\nu}_1 \frac{e^{-it}}{(1+r^2)^{1/2}}}$$

(C.I., P. Sundell)

Fronsdal field modes

- The electric field strength components of the spin-1 massless field follow from the energy-2, spin-1 element

$$\Phi' \;=\; \Phi'_{2;(1)0r} \;\propto\; M_{or}e^{-4E} \;\propto\; (\sigma_{0r})^{\alpha\beta}\frac{\partial^2}{\partial\chi^\alpha\partial\chi^\beta}e^{-4E+\chi y+\bar{\chi}\bar{y}}\Big|_{\chi=\bar{\chi}=0} + \text{h.c.}$$

  and, more generally, the generating function of arbitrary spin-s massless particle modes admits a contour integral presentation with integrand given by

$$\Phi'(Y,\eta) \;=\; \Phi'_{\rm pt}(Y,\eta) \;=\; e^{-4\eta E+\chi y+\bar{\chi}\bar{y}}$$

  (different spin-s modes singled out by specific projections of an even number ≥ 2s of χ and $\bar{\chi}$ derivatives and an appropriate contour integration $\mathcal{O}_s$ on *η*).

- One can check that, in *L*-gauge, the resulting generating function

$$\Phi^{(L)}_{\rm pt}(x,Y) \;=\; \mathcal{O}\,\frac{1-x^2}{1-2i\eta x_0+\eta^2x^2}\exp\left[iyM\bar{y}+\bar{\chi}(\bar{P}\bar{y}-i\bar{x}A^{-1}\chi)-\chi A^{-1}y\right]$$

  gives rise to the correct expression for the various modes (*M*, *P* and *A* → *x*- and *η*-dependent matrices).

- What about the gauge fields? More precisely, where does the COMST sit in the gauge function method?

Z-space connection for particle modes

- To answer, it is instructive to examine the *Z*-space connection at first order:

$$V^{(L)(1)}_{\mathrm{pt}\,\alpha}(x,Y,z) \;=\; \mathcal{O}L^{-1}\star\widetilde{V}^{(1)}_{\mathrm{pt}\,\alpha}(\eta,Y,z)\star L \;=\; \Psi^{(L)}_{\mathrm{pt}}\star V^{(1)}_{\alpha}(z)$$

$$=\;\mathcal{O}\,\frac{1-x^2}{1-2i\eta x_0+\eta^2x^2}\,\frac{(D\widetilde{y})_\alpha}{\widetilde{y}D\widetilde{y}}\,e^{i\widetilde{y}z+\bar\chi(\bar P\bar y-i\bar x A^{-1}\chi)}$$

$$\widetilde{y}_\alpha \;:=\; y_\alpha+M_\alpha{}^{\dot\beta}(\eta,x)\bar y_{\dot\beta}+iA^{-1}_\alpha{}^{\beta}(\eta,x)\chi_\beta\,,\qquad \frac{1}{2i\widetilde{y}D\widetilde{y}}\;=\;\int_{-1}^{1}\frac{dt}{(t-1)^2}\;e^{-\frac{i}{2}\frac{t+1}{t-1}\widetilde{y}D\widetilde{y}}$$

→ pole at a plane $\widetilde{y}$ = 0! Needs to be understood.

- Such singular behaviour goes hand in hand with the peculiar feature of *V* in *L*-gauge, that solves the deformed oscillator problem

$$\partial_{[\alpha}\widehat{V}^{(L)(1)}_{\mathrm{pt}\,\beta]}\;=\;-\frac{i}{4}\,\epsilon_{\alpha\beta}\,b\Psi^{(L)}_{\mathrm{pt}}\star\kappa_z$$

and is at the same time covariantly constant wrt the AdS covariant derivative,

$$D^{(0)}\widehat{V}^{(L)(1)}_{\mathrm{pt}\,\alpha}\;=\;0$$

- The latter feature implies $$d_Z\widehat{W}^{(L)(1)}_{\mathrm{pt}}\;=\;0$$

and indeed remains $W= W^{(0)}=L^{-1}\star dL$ to all orders, in *L*-gauge. COMST??

Vasiliev gauge

- Clearly, in order to compare with standard perturbation theory and read off Fronsdal fields explicitly in *W* we have to modify the gauge function in such a way that the local dof in $\Phi^{(1)}$ stay untouched but $V^{(1)}$ is brought to the Vasiliev gauge,

$$\widehat{V}_\alpha^{(L)(1)} \to \widehat{V}_\alpha^{(G)(1)} \quad \text{such that} \quad z^\alpha \widehat{V}_\alpha^{(G)(1)} \;=\; 0$$

- This can be achieved, within the gauge function method, by choosing a more complicated, field-dependent gauge function G = L ★ H, H=H(x,Y,Z),

$$\boxed{\begin{aligned} \widehat{\Phi}^{(G)} &= \widehat{G}^{-1} \star \Phi' \star \pi(\widehat{G}) \;=\; \Phi^{(L)} + \text{h.o.t.}\;, \\ \widehat{S}_\alpha^{(G)} &= \widehat{G}^{-1} \star \widehat{S}'_\alpha \star \widehat{G} \;=\; z_\alpha + \widehat{V}_\alpha^{(L)(1)} + \partial_\alpha \widehat{H}^{(1)} + \text{h.o.t.}\;, \\ \widehat{W}^{(G)} &= \widehat{G}^{-1} \star d\widehat{G} \;=\; L^{-1} \star dL + D^{(0)} \widehat{H}^{(1)} + \text{h.o.t.} \end{aligned}}$$

$$\widehat{V}_\alpha^{(G)(1)} - \widehat{V}_\alpha^{(L)(1)} \;=\; \partial_\alpha \widehat{H}^{(1)}$$

- Integrating this eq. (using Vasiliev gauge condition and the regularity in *z* of $V^{(L)}$) →

$$H^{(1)} \;=\; \mathcal{O}\, \frac{1-x^2}{1-2i\eta x_0+\eta^2 x^2}\, \frac{zD\widetilde{y}}{\widetilde{y}D\widetilde{y}}\, \frac{e^{i\widetilde{yz}}-1}{i\widetilde{y}z}\, \exp\left[\bar{\chi}(\bar{P}\bar{y} - i\bar{x}A^{-1}\chi)\right] + H_0^{(1)}(Y) + \text{h.c.}$$

→ well-behaved at the spacetime boundary and in Z, but inherits the pole in Y. (However, its contribution to *W*, $D^{(0)}H^{(1)}$, in Z=0 is *regular!*)

Vasiliev gauge

- Let's take a look at the transformed master fields.
- The transformed Z-space connection is regular everywhere, and coincides with the form coming from the usual perturbation theory,

$$\widehat{V}_{\alpha}^{(G)(1)} = z_{\alpha}\mathcal{O}\,\frac{1-x^2}{1-2i\eta x_0+\eta^2x^2}\,\frac{e^{i\tilde{y}z}(1-i\tilde{y}z)-1}{(\tilde{y}z)^2}\,e^{\bar{\chi}(\bar{P}\bar{y}-i\bar{x}A^{-1}\chi)} = z_{\alpha}\int_0^1 dt\,t\,\Phi(-tz,\bar{y})e^{ity^{\alpha}z_{\alpha}}$$

- This means that the usual generating function of gauge fields will also be regular and give rise to the usual COMST, since

$$\partial_{\alpha}\widehat{W}^{(G,1)} = \partial_{\alpha}(D^{(0)}\widehat{H}^{(1)}) = D^{(0)}(\partial_{\alpha}\widehat{H}^{(1)}) = D^{(0)}(\widehat{V}_{\alpha}^{(G,1)}-\widehat{V}_{\alpha}^{(L,1)}) = D^{(0)}\widehat{V}_{\alpha}^{(G,1)}$$

→ the Z-dependent part of W is real-analytic in all variables and coincides with the usual solution $\rho\, D^{(0)}\rho\, (\Phi\star J)$.

- It is interesting to understand precisely how this happens from a singular gauge function:

$$\begin{aligned}\widehat{W}^{(G,1)} &= D^{(0)}\widehat{H}^{(1)} = D^{(0)}\widehat{H}_0^{(1)} + D^{(0)}\widehat{H}_p^{(1)}\\ &= D^{(0)}\widehat{H}_0^{(1)} + \underbrace{\left(D^{(0)}\widehat{H}_p^{(1)}\right)_{Z=0}}_{\omega_{\mathrm{reg}}(Y)+\omega_{\mathrm{irreg}}(Y)} + \underbrace{D^{(0)}\widehat{H}_p^{(1)} - \left(D^{(0)}\widehat{H}_p^{(1)}\right)_{Z=0}}_{=-\rho D^{(0)}\rho(\Phi\star J)}\end{aligned}$$

Vasiliev gauge

- But ω_{irreg} has no impact on the gauge field equations.
- Indeed, acting with $D^{(0)}$ on $W|_{Z=0}$ selects the usual r.h.s. of the COMST:

$$\widehat{W}^{(G,1)}|_{Z=0} = D^{(0)} H_0^{(1)} + i\,\Omega^{\underline{\alpha\beta}}\partial^{(Y)}_{\underline{\alpha}}\widehat{V}^{(L,1)}_{\underline{\beta}}\Big|_{Z=0}$$

$$D^{(0)}(\widehat{W}^{(G,1)}|_{Z=0}) \;=\; iD^{(0)}\left(\Omega^{\underline{\alpha\beta}}\partial^{(Y)}_{\underline{\alpha}}V^{(L,1)}_{\underline{\beta}}\Big|_{Z=0}\right) \;=\; -\Omega^{\underline{\alpha\beta}}\wedge\Omega^{\underline{\gamma\delta}}\,\partial^{(Y)}_{\underline{\alpha}}\;\partial^{(Y)}_{\underline{\gamma}}\;\underbrace{\left(\partial^{(Z)}_{[\underline{\delta}}\widehat{V}^{(L,1)}_{\underline{\beta}]}\right)\Big|_{Z=0}}_{\propto\;\epsilon_{\alpha\beta}\Phi(0,\bar{y})+\mathrm{c.c.}}$$

- Any irregular term in Y inside $P := (D^{(0)}H_p^{(1)})_{Z=0}$ is (locally) exact, since

$$P(Y) \;=\; \sum_{n\in\mathbb{Z}} P_n\;, \qquad (E_Y - n)P_n \;=\; 0\;, \quad E_Y := Y^{\underline{\alpha}}\partial^{(Y)}_{\underline{\alpha}}$$

$$(E_Y D^{(0)} - D^{(0)}E_Y)f(x,dx,Y) = 0 \quad\Rightarrow\quad (E_Y - n)D^{(0)}P_n \;=\; 0$$

- But $D^{(0)}P$ is real-analytic → $\quad D^{(0)}P_n \;=\; 0, \quad n<0 \quad\Rightarrow\quad P_{n<0} \;=\; D^{(0)}Q_{n<0}$

→ can fix $\quad H_0^{(1)} \;=\; -\sum_{n<0} Q_n \quad\Rightarrow\quad \widehat{W}^{(1)} \;=\; \omega_{\text{reg}}(Y) - \rho D^{(0)}\rho(\Phi\star J)$

Singular gauge functions and L-gauge

- What is the actual meaning of those singularities of the gauge function?

- It is not an accident of having started from a weird gauge. In fact, in any solution

$$D^{(0)}\widehat{W}^{(G,1)} = 0 \qquad \widehat{W}^{(G,1)} = D^{(0)}\widehat{\xi}^{(1)}$$

that gives rise to the COMST $\widehat{W}^{(G,1)} = D^{(0)}\widehat{\xi}^{(1)} = \omega(Y) - \rho D^{(0)}\widehat{V}^{(G,1)}$

the gauge function ξ can only differ from the above-found singular solution $H^{(1)} = H_p^{(1)} - Q$ by a quantity $\Delta H^{(1)}$ such that

1) $D^{(0)}\Delta H^{(1)} = 0$ (not to change the gauge fields), and

2) $E_Z\,\Delta H^{(1)} = 0$ (not to alter the Vasiliev gauge condition)

→ $\Delta H^{(1)}$ cannot eat up the singularities of neither $H_p^{(1)}$ nor Q ,
ξ is as singular as $H^{(1)}$.