Homotopy operators. Properties and application in HS equations

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Outline

Introduction

Dynamic HS equations schematically

$$
\begin{aligned}
&\mathrm{d}_x\omega + \omega * \omega = \Upsilon(\omega,\omega,C) + \Upsilon(\omega,\omega,C,C) + \dots\,, \\
&\mathrm{d}_x C + [\omega, C]_* = \Upsilon(\omega, C, C) + \dots
\end{aligned}
$$

$\omega = \mathrm{d}x^\mu \omega_\mu\,(Y,K|x)$ – generating function for HS gauge fields.

$C = C\,(Y,K|x)$ – generating function for HS stress-tensors (Maxwell, Weyl, etc) + scalar $C\,(0,K|x)$.

$\Upsilon(\omega,\omega,C,C), \Upsilon(\omega,C,C), \dots$ – might contain infinite number of derivatives between C-fields.

Υ-vertices can be obtained from generating system with doubled amount of spinorial variables

Introduction

Vasiliev equations in $d = 4$

$$
\begin{aligned}
&\mathrm{d}_x W + W * W = 0\,,\\
&\mathrm{d}_x S + [W, S]_* = 0\,,\\
&\mathrm{d}_x B + [W, B]_* = 0\,,\\
&S * S = i(\theta^A \theta_A + \eta B * \gamma + \bar{\eta} B * \bar{\gamma})\,,\\
&[S, B]_* = 0\,.
\end{aligned}
$$

Here $W = \mathrm{d}x^\mu W_\mu\,(Z, Y, K)$, $S = \theta^A S_A\,(Z, Y, K)$, $B = B\,(Z, Y, K)$.
$Z_A = (z_\alpha, \bar{z}_{\dot{\alpha}})$, $Y_A = (y_\alpha, \bar{y}_{\dot{\alpha}})$, $\theta^A = \left(\theta^\alpha, \bar{\theta}^{\dot{\alpha}}\right)$, $K = \left(k, \bar{k}\right)$

$$\{y_\alpha, k\} = \{z_\alpha, k\} = \{\theta^\alpha, k\} = 0, \;\; [\bar{y}_{\dot{\alpha}}, k] = [\bar{z}_{\dot{\alpha}}, k] = [\bar{\theta}^{\dot{\alpha}}, k] = 0, \;\; k^2 = 1.$$

$$\mathrm{d}_x = \mathrm{d}x^\mu \frac{\partial}{\partial x^\mu}, \; \{\theta^A, \theta^B\} = \{\mathrm{d}x^\mu, \theta^A\} = 0 \implies \{\frac{\partial}{\partial \theta^A}, \theta^B\} = \{\frac{\partial}{\partial \theta^A}, \mathrm{d}x^\mu\} = 0.$$

Introduction

Star-product

$$(f * g)(Z,Y) = \frac{1}{(2\pi)^4} \int dU dV f(Z+U, Y+U) g(Z-V, Y+V) e^{iU_A V^A} .$$

$$[Y_A, Y_B]_* = -[Z_A, Z_B]_* = 2i\epsilon_{AB} , \qquad [Y_A, Z_B]_* = 0 .$$

Central elements $\gamma, \bar{\gamma}$

$$\gamma = e^{iz_\alpha y^\alpha} k \theta^\alpha \theta_\alpha , \qquad \bar{\gamma} = e^{i\bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}}} \bar{k} \bar{\theta}^{\dot{\alpha}} \bar{\theta}_{\dot{\alpha}}$$

Inner Klein operators

$$f(z, \bar{z}, y, \bar{y}) * e^{iz_\alpha y^\alpha} = e^{iz_\alpha y^\alpha} * f(-z, \bar{z}, -y, \bar{y}) .$$

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Perturbative expansion

$$B_0 = 0\,, \quad S_0 = \theta^A Z_A\,, \quad W_0 = \omega\left(y, \bar{y}\right).$$

Here $\omega\left(y, \bar{y}\right)$ satisfies $\mathrm{d}_x \omega + \omega * \omega = 0$.

$$[S, B]_* = 0 \implies [S_0, B_1]_* + [S_1, B_0]_* = 0,$$

Using star-product $[Z_A, f\left(Z, Y\right)]_* = -2i \frac{\partial}{\partial Z^A} f\left(Z, Y\right)$

$$[S_0, B_1] = -2i\theta^A \frac{\partial}{\partial Z^A} B_1\left(Z, Y, K\right) = -2i\mathrm{d}_Z B_1\left(Z, Y, K\right) = 0.$$

$$B_1\left(Z, Y, K\right) = C\left(Y, K\right).$$

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$$-2i\mathrm{d}_Z S_1 = i\eta C * \gamma + i\bar{\eta} C * \bar{\gamma}$$

Typical equation

$$\mathrm{d}_Z f\left(Z, Y|\theta\right) = J\left(Z, Y|\theta\right)$$

Solution up to d_Z-exact terms is given by homotopy operator $\triangle_Q$

$$f\left(Z, Y|\theta\right) = \triangle_Q J\left(Z, Y|\theta\right),$$

where

$$\triangle_Q J\left(Z, Y|\theta\right) \equiv \left(Z+Q\right)^A \frac{\partial}{\partial\theta^A} \int_0^1 \frac{dt}{t} J\left(tZ - \left(1-t\right)Q, Y|t\theta\right).$$

Where Q should be Z-independent. When $Q=0$, $\triangle_0$ – conventional homotopy.

Homotopy operator $\triangle_Q$

$$\triangle_Q J\left(Z,Y|\theta\right) \equiv \left(Z+Q\right)^A \frac{\partial}{\partial\theta^A}\int_0^1 \frac{dt}{t} J\left(tZ-\left(1-t\right)Q, Y|t\theta\right).$$

How to obtain this formula?

From Z to $Z'_A = Z_A + Q_A$

$$J\left(Z,Y|\theta\right) \Longrightarrow J\left(Z'-Q, Y|\theta\right)$$

Conventional homotopy over Z'

$$Z'^A \frac{\partial}{\partial\theta^A}\int_0^1 \frac{dt}{t} J\left(tZ'-Q, Y|t\theta\right).$$

Homotopy operator properties

Index A in $\left(Z_A, Y_A, \theta^A\right)$ takes arbitrary values

Resolution of identity

$$\mathrm{d}_Z \triangle_Q + \triangle_Q \mathrm{d}_Z + h_Q = 1, \quad \text{where} \quad h_Q f\left(Z, Y|\theta\right) \equiv f\left(-Q, Y|0\right)$$

Using definition and resolution of identity one cas show

$$\begin{aligned} &\triangle_Q \triangle_P + \triangle_P \triangle_Q = 0 \implies \triangle_P \triangle_P = 0, \\ &h_P \triangle_Q = -h_Q \triangle_P \quad \implies h_P \triangle_P = 0, \\ &\triangle_B - \triangle_A = \{\mathrm{d}_Z, \triangle_A \triangle_B\} + h_A \triangle_B \end{aligned}$$

Moreover

$$\triangle(\rho) := \int dQ \rho(Q) \triangle_Q, \quad \text{with} \quad \int dQ \rho(Q) = 1.$$

Homotopy operator properties

Two component relations $\left(Z_A, Y_A, \theta^A\right) \Longrightarrow (z_\alpha, y_\alpha, \theta^\alpha)$

$$\triangle_b\triangle_a f(z,y)\theta^\beta\theta_\beta = 2\int_{[0,1]^3} d^3\tau\,\delta(1-\tau_1-\tau_2-\tau_3)(z+b)_\alpha(z+a)^\alpha f(\tau_1 z-\tau_3 b-\tau_2 a, y)$$

$$h_c\triangle_b\triangle_a f(z,y)\theta^\beta\theta_\beta = 2\int_{[0,1]^3} d^3\tau\,\delta(1-\tau_1-\tau_2-\tau_3)(b-c)_\gamma(a-c)^\gamma f(-\tau_1 c-\tau_3 b-\tau_2 a, y)\,.$$

As consequence

$$h_{(\mu+1)q_2-\mu q_1}\triangle_{q_2}\triangle_{q_1} = 0 \;\; \forall\mu\in\mathbb{C}$$

$$\triangle_b\triangle_a\gamma = 2\int_{[0,1]^3} d^3\tau\,\delta(1-\tau_1-\tau_2-\tau_3)(z+b)_\alpha(z+a)^\alpha e^{i(\tau_1 z-\tau_2 a-\tau_3 b)_\beta y^\beta}k,$$

$$h_c\triangle_b\triangle_a\gamma = 2\int_{[0,1]^3} d^3\tau\,\delta(1-\tau_1-\tau_2-\tau_3)(b-c)_\beta(a-c)^\beta e^{-i(\tau_1 c+\tau_2 a+\tau_3 b)_\alpha y^\alpha}k.$$

As consequence

$$h_{a+\alpha y}\triangle_{b+\alpha y}\triangle_{c+\alpha y}\gamma = h_a\triangle_b\triangle_c\gamma \;\; \forall\alpha\in\mathbb{C}\,.$$

Homotopy operator properties

$$(\triangle_d - \triangle_c)(\triangle_a - \triangle_b)\gamma = (h_d - h_c)\triangle_a\triangle_b\gamma\,,$$

Star-exchange properties

$$\triangle_{q+\alpha y}(C(y,k) * \Gamma(z,y|\theta)) = C(y,k) * \triangle_{q+(1-\alpha)p+\alpha y}\Gamma(z,y|\theta)\,,$$

where q is z and y independent spinor.

$$p_\alpha = -i\partial^C_\alpha\,, \quad p_\alpha C(y,k) \equiv C(y,k)p_\alpha = -i\frac{\partial}{\partial y^\alpha}C(y,k)$$

$$\triangle_{q+\alpha y}(\Gamma(z,y;\theta) * C(y,k)) = \triangle_{q+(1+\alpha)p+\alpha y}\Gamma(z,y;\theta) * C(y,k)$$

$$\triangle_{q+\alpha y}\gamma * C(y,k) = C(y,k) * \triangle_{q+2p+\alpha y}\gamma\,.$$

Central on-mass-shell theorem. Preliminaries

Only holomorphic sector

$$-2i\mathrm{d}_z S_1 = i\eta C * \gamma \implies S_1 = -\frac{\eta}{2}\triangle_0 \left(C * \gamma\right) = -\frac{\eta}{2} C * \triangle_p \left(\gamma\right),$$

where $d_z = \theta^\alpha \dfrac{\partial}{\partial z^\alpha}$.

$$2id_z W_1 = \mathrm{d}_x S_1 + [\omega, S_1]_* \implies W_1 = \frac{1}{2i}\triangle_0 \left(\mathrm{d}_x S_1 + [\omega, S_1]_*\right)$$

$$W_1 = -\frac{\eta}{4i}\left(C * \omega * \triangle_{p+t}\triangle_{p+2t}\gamma - \omega * C * \triangle_{p+t}\triangle_p \gamma\right),$$

where

$$t_\alpha = -i\partial^\omega_\alpha, \quad t_\alpha \omega(y) \equiv \omega(y) t_\alpha = -i\frac{\partial}{\partial y^\alpha}\omega(y).$$

Thanks to star-exchange formulas!

Central on-mass-shell theorem. Dynamic equations

$$\mathrm{d}_x\omega + \omega * \omega = -\underbrace{\mathrm{d}_x W_1} - \omega * W_1 - W_1 * \omega$$

$\mathrm{d}_x W_1$ is proportional to z_α.

$$\mathrm{d}_x\omega + \omega * \omega = 0, \quad \mathrm{d}_x C + \omega * C - C * \omega = 0.$$

$$\mathrm{d}_x \left(\omega * C * \triangle_{p+t}\triangle_p\gamma\right) = -\omega * \omega * C * \triangle_{p+t_1+t_2}\triangle_p\gamma + \dots$$

$$\begin{aligned}\mathrm{d}_x\omega + \omega * \omega = \frac{\eta}{4i}\omega * \omega * C * \left(\triangle_{p+t_1+t_2} - \triangle_{p+t_2}\right)\left(\triangle_p - \triangle_{p+t_2}\right)\gamma + \\ + \frac{\eta}{4i}\omega * C * \omega * \left(\triangle_{p+t_1+t_2} - \triangle_{p+2t_2}\right)\left(\triangle_{p+t_2} - \triangle_{p+t_1+2t_2}\right)\gamma + \\ + \frac{\eta}{4i}C * \omega * \omega * \left(\triangle_{p+t_1+t_2} - \triangle_{p+t_1+2t_2}\right)\left(\triangle_{p+t_1+2t_2} - \triangle_{p+2t_1+2t_2}\right)\gamma.\end{aligned}$$

Recall the following property

$$(\triangle_d - \triangle_c)(\triangle_a - \triangle_b)\gamma = (h_d - h_c)\triangle_a\triangle_b\gamma\,.$$

Central on-mass-shell theorem. Dynamic equations

$$\mathrm{d}_x\omega + \omega * \omega = \Upsilon_{\omega\omega C} + \Upsilon_{\omega C\omega} + \Upsilon_{C\omega\omega}$$

$$\begin{aligned}\Upsilon_{\omega\omega C} &= \frac{\eta}{4i}\omega * \omega * C * h_{p+t_1+t_2}\triangle_p\triangle_{p+t_2}\gamma = \\ &= \frac{\eta}{2i}\int_{[0,1]^3} d^3\tau\delta(1-\tau_1-\tau_2-\tau_3)e^{i(1-\tau_3)\partial_1^\alpha\partial_{2\alpha}}\times \\ &\times \partial^\alpha\omega((1-\tau_1)y,\bar{y})\,\bar{*}\,\partial_\alpha\omega(\tau_2 y,\bar{y})\,\bar{*}\,C(\underbrace{-\tau_1\partial_1-(1-\tau_2)\partial_2},\bar{y};K)k\end{aligned}$$

No y-depence in C! Ultra-local

Conclusion

- Background independent central on-mass-shell theorem with ultra-local structure of vertices
- Thanks to star-exchange formulas structures $\triangle\gamma$, $\triangle\triangle\gamma$, $\triangle(\triangle\gamma * \triangle\gamma)$, etc. appeared to be the only thing to be analyzed
- Generalized homotopy operators may be also applied in $3d$ Prokushkin-Vasiliev theory (work in progress)