On dual description of η-deformed integrable sigma models

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- Studying QFT's in the strong coupling regime is an interesting problem

- Analysis simplifies if the theory shares the property of the so called weak/strong coupling duality

- Among known examples: Coleman-Mandelstam duality, electric magnetic duality, various string dualities, AdS/CFT duality etc

- Today, we will study another example of this phenomenon. Pair of dual theories is: the so called η-deformed symmetric space sigma model and certain Toda theory.

- Both theories are two-dimensional integrable QFT's.

- We start with an example: $O(3)$ sigma-model

$$S = \frac{1}{g^2} \int \Big((\partial_\mu \varphi_1)^2 + (\partial_\mu \varphi_2)^2 + (\partial_\mu \varphi_3)^2 \Big) d^2x, \qquad \sum_{k=1}^{3} \varphi_k^2 = 1.$$

- This theory is conformal and integrable at the classical level (Pohlmeyer).

- As QFT it corresponds to an asymptotically free theory with a dynamically generated mass scale and is also intregrable (Polyakov).

- Its $S-$matrix is constraint by integrability (Zamolodchikov$\times$2)

$$S_{ij}^{kl}(\theta) = \delta_{ij}\delta_{kl} S_1(\theta) + \delta_{ik}\delta_{jl} S_2(\theta) + \delta_{il}\delta_{jk} S_3(\theta),$$

$$S_1(\theta) = \frac{2i\pi\theta}{(\theta + i\pi)(\theta - 2i\pi)}, \quad S_2(\theta) = \frac{\theta(\theta - i\pi)}{(\theta + i\pi)(\theta - 2i\pi)}, \quad S_3(\theta) = \dots$$

where $\theta = \theta_1 - \theta_2$ is the rapidity difference of the incoming states.

- This rational $S-$matrix corresponds to the Yangian $Y(SO(3))$. Yangians always admit one-parametric deformation called the quantum affine group.

- Corresponding $S-$matrix is a trigonometric one
$$S_{++}^{++}(\theta) = \frac{\sinh\lambda(\theta - i\pi)}{\sinh\lambda(\theta + i\pi)}, \quad S_{+0}^{+0} = \frac{\sinh\lambda\theta}{\sinh\lambda(\theta - 2i\pi)} S_{++}^{++}(\theta), \ldots$$
where λ is the deformation parameter. At $\lambda \to 0$ we are back to the rational case.

- It is naturally to expect that there should exist an integrable deformation of the $O(3)$ SM, which corresponds to this trigonometric $S-$matrix.

- At least this deformed sigma-model should correspond to renormalizable QFT.

- Analysis of general sigma-model

$$\mathcal{A} = \frac{1}{4\pi}\int \left(G_{\mu\nu}(\varphi)\partial\varphi^{\mu}\bar{\partial}\varphi^{\nu} + \ldots\right) d^2x,$$

requires the target space metric $G_{\mu\nu}$ to satisfy the Ricci flow equations

$$R_{\mu\nu} + \cdots = -\dot{G}_{\mu\nu}, \quad \text{where} \quad \cdot = \frac{d}{dt}, \quad t \sim \log\frac{\Lambda^*}{\Lambda}$$

- There is remarkable solution in 2d (Fateev, Onofri, Zamolodchikov)

$$ds^2 = \frac{\kappa}{\nu}\left(\frac{d\zeta^2}{(1-\zeta^2)(1-\kappa^2\zeta^2)} + \frac{(1-\zeta^2)d\phi^2}{(1-\kappa^2\zeta^2)}\right), \quad \kappa = -\tanh\nu t$$

- While embedded in 3d it looks like a "sausage" of length $L = -\sqrt{\nu}t$: infinitely long in the UV ($t \to -\infty$) and shrinking at the intermediate scale ($t \sim 0$). At $\nu \to 0$ we are back to the round sphere.

- It has been conjectured that the full theory promoted by the one-loop action is integrable with the trigonometric $S-$matrix.

- We note that

$$S^{++}_{++}(\theta)=\frac{\sinh\lambda(\theta-i\pi)}{\sinh\lambda(\theta+i\pi)},\qquad S^{+0}_{+0}=\frac{\sinh\lambda\theta}{\sinh\lambda(\theta-2i\pi)}S^{++}_{++}(\theta),\ldots$$

simplifies in the limit $\lambda\to\frac{1}{2}$: $S^{++}_{++}(\theta)\to -1$ ...

- It looks like in the limit we have a theory of free boson and Dirac fermion of the same mass.

- Away from $\lambda=\frac{1}{2}$ one expects some local QFT. Remarkably, it has been guessed by Alyosha Zamolodchikov

$$\mathcal{L}=\frac{1}{8\pi}\Big(\partial_\mu\Phi\Big)^2+i\bar\psi\gamma^\mu\partial_\mu\psi+\frac{\pi b^2}{2(1+b^2)}\Big(\bar\psi\gamma^\mu\psi\Big)^2-$$

$$-m\bar\psi\psi\cosh(b\Phi)-\frac{m^2}{8\pi b^2}\sinh^2(b\Phi),$$

provided that $\lambda=\frac{1}{2(1+b^2)}$.

- As it becomes clear recently the sausage sigma-model belongs to the more general class of the so called $\eta-$deformed sigma models.

- It's all started with the seminal Klimciks observation (Klimcik 2008). First, one deforms the PCF model

$$\mathcal{S} = \frac{1}{2}\int \text{Tr}\left(\mathbf{g}^{-1}\partial_+\mathbf{g}\mathbf{g}^{-1}\partial_-\mathbf{g}\right) d^2x.$$

This theory has a global $G_L \times G_R$ symmetry which acts as

$$\mathbf{g} \to U\mathbf{g}V, \qquad U, V \in G.$$

This theory is known to be classically/quantum integrable (Zakharov-Mikhailov, Polyakov-Wiegmann).

- Now, let H be the Lie subgroup of G and $\mathfrak{h} = \text{Lie}(H)$. We can obtain the sigma-model on a quotient space G/H by gauging

$$\partial_\pm \to D_\pm = \partial_\pm - A_\pm, \quad A_\pm \in \mathfrak{h}.$$

The non-trivial fact is that it preserves integrability (classically!).

- The deformation is given in terms of the linear operator $\mathcal{R}$: $\mathfrak{g} \to \mathfrak{g}$

$$\langle a, \mathcal{R}b \rangle = -\langle \mathcal{R}a, b \rangle,$$

and satisfies

$$[\mathcal{R}a, \mathcal{R}b] + \mathcal{R}([a, \mathcal{R}b] + [\mathcal{R}a, b]) - [a, b] = 0,$$

called the modified Yang-Baxter equation.

- With the operator $\mathcal{R}$ at hand we define the deformed PCF action as

$$\mathcal{S} = \frac{1}{2} \int \mathrm{Tr} \left(\mathbf{g}^{-1} \partial_+ \mathbf{g} \, \frac{1}{1 - \eta \mathcal{R}} \, \mathbf{g}^{-1} \partial_- \mathbf{g} \right) d^2 x,$$

where η is the deformation parameter ($\eta = i\kappa$). It has been shown by Klimcik that this particular type of deformations survives the integrability.

- Note that the deformed theory is still left G invariant. So, one can apply the coset construction.

- The action of the $\eta-$deformed coset sigma model can be written in the form (Delduc et all)

$$\mathcal{S} = \frac{1}{2}\int \mathsf{Tr}\left(\left(\mathbf{g}\partial_{+}\mathbf{g}^{-1}\right)^{(\mathsf{c})} \frac{1}{1-\eta\mathcal{R}_{\mathfrak{g}}\circ \mathsf{P}_{\mathsf{C}}} \left(\mathbf{g}\partial_{-}\mathbf{g}^{-1}\right)^{(\mathsf{c})}\right) d^2x,$$

where $\mathcal{R}_{\mathfrak{g}} = \mathsf{Ad}\,\mathfrak{g}^{-1}\circ\mathcal{R}\circ\mathsf{Ad}\,\mathfrak{g}$ and P_{C} is the projection on the coset space.

- It has been shown that this theory is integrable provided that G/H is a symmetric space.

- For example $G/H = SO(N)/SO(N-1)$ provides an integrable deformation of the $O(N)$ sigma-model.

- As expected $SO(3)/SO(2)$ case corresponds to the deformed $O(3)$ sigma-model and coincides with the sausage model for certain choice of the operator $\mathcal{R}$.

- One way to explain the duality requires bosonization. Namely, we rewrite

$$\mathcal{L}=\frac{1}{8\pi}\Big(\partial_\mu\Phi\Big)^2+i\bar{\psi}\gamma^\mu\partial_\mu\psi+\frac{\pi b^2}{2(1+b^2)}\Big(\bar{\psi}\gamma^\mu\psi\Big)^2-m\bar{\psi}\psi\cosh(b\Phi)+\dots$$

  in terms of two bosonic fields (here $\beta=\sqrt{1+b^2}$)

$$\mathcal{L}=\frac{1}{8\pi}\Big(\partial_\mu\varphi\Big)^2+\frac{1}{8\pi}\Big(\partial_\mu\Phi\Big)^2-m\cos(\beta\phi)\cosh(b\Phi)+\dots.$$

- This theory is integrable $[\mathbf{I}_{2k-1},\mathbf{I}_{2l-1}]=0$. Perturbatively

$$\mathbf{I}_{2k-1}=\mathbf{I}_{2k-1}^{\text{free}}+O(m),\quad \text{with}\quad \mathbf{I}_{2k-1}^{\text{free}}=\frac{1}{2\pi}\int_{\mathcal{C}}W_{2k}(\partial\varphi,\partial\Phi)dz,$$

- For example,

$$W_2(\partial\varphi,\partial\Phi)=(\partial\varphi)^2+(\partial\Phi)^2.$$

- Higher-spin densities can be defined from the requirement that they commute with the perturbation

$$[\mathbf{I}_{2k-1}^{\text{free}},\int e^{\pm i\beta\varphi\pm b\Phi}dz]=0.$$

- Motivated by the explicit form of the bosonic Lagrangian we will study more general setting. Let $\varphi = (\varphi_1, \ldots, \varphi_{N-1})$ be the $N-1$ component bosonic field and consider the theory

$$\mathcal{L} = \frac{1}{8\pi}\Big(\partial_\mu \varphi\Big)^2 + \Lambda \sum_{r=1}^{N} e^{(\boldsymbol{\alpha}_r, \varphi)},$$

where $(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N)$ is a given set of vectors, which is required to have maximal rank.

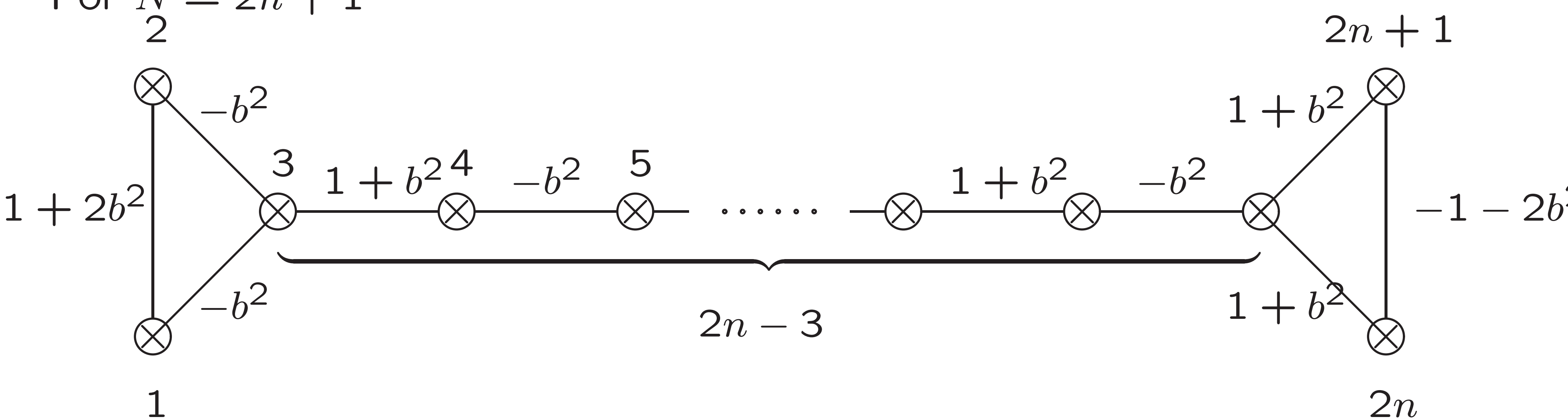
- We are only interested in the quantum field theories of this form with infinitely many integrals of motion. In the leading order in Λ this constraints the integrals of motion $\mathbf{I}_k^{\text{free}}$ to obey

$$[\mathbf{I}_k^{\text{free}}, \int e^{(\boldsymbol{\alpha}_r, \varphi)} dz] = 0, \quad \text{for all} \quad r = 1, \ldots, N.$$

- Practically, we study the equations
$$\oint_{\mathcal{C}_z} e^{(\mathbf{a}_r,\varphi(\xi))} W_{k-1}(z)d\xi = \partial(\dots),$$
where $W_k(z)$ is some preferred set of currents in $W-$algebra. This implies that the charges
$$\mathbf{I}_{k-1} = \int W_k(z)dz,$$
commute with all exponentials.

- There many solutions. Important for us, correspond to fermionic screenings: $(\boldsymbol{\alpha},\boldsymbol{\alpha}) = -1$

For $N = 2n+1$



This picture defines an integrable system $[\mathbf{I}_{2k-1}, \mathbf{I}_{2l-1}] = 0$ with

$$\mathbf{I}_{2k-1} = \int W_{2k}(x)dx.$$

Explicitly one has $\varphi = (\Phi_1, \ldots, \Phi_n, \phi_1, \ldots, \phi_n)$

$$W_2(x) = (\partial\Phi, \partial\Phi) + (\partial\phi, \partial\phi),$$

and

$$W_4(x) = ((\partial\Phi, \partial\Phi) + (\partial\phi, \partial\phi))^2 + \frac{2n-1}{3}\sum_{k=1}^{n}\Big(\frac{1}{b^2}(\partial\Phi_k)^4 + \frac{1}{a^2}(\partial\phi_k)^4\Big) +$$
$$+2(2n-1)\sum_{k=1}^{n}\Big((\partial\Phi_k)^2+(\partial\phi_k)^2\Big)\Bigg(\frac{1}{a}\sum_{j>k}\partial^2\Phi_j+\frac{1}{b}\sum_{j<k}\partial^2\phi_j-\frac{2}{2n-1}\sum_{j=1}^{n}(j-1)\Bigg(\frac{1}{a}\partial$$
$$+\Bigg(\frac{4(n+1)}{3}+\frac{2n-1}{3}\Big(\frac{1}{b^2}+\frac{2}{a^2}\Big)\Bigg)(\partial^2\Phi, \partial^2\Phi)+\Bigg(\frac{4(n+1)}{3}+\frac{2n-1}{3}\Big(\frac{2}{b^2}+\frac{1}{a^2}\Big)\Bigg)(\partial^2$$
$$+2\sum_{i\leq j}(i-1)(2(j-n)-1)\Big(2-\delta_{ij}\Big)\Big(\frac{1}{a^2}\partial^2\Phi_i\partial^2\Phi_j + \frac{1}{b^2}\partial^2\phi_{n-i+1}\partial^2\phi_{n-j+1}\Big)+$$
$$+\frac{2}{ab}\Big(4\sum_{i,j}(i-1)(n-j)\partial^2\Phi_i\partial^2\phi_j - (2n-1)\sum_{i>j}(2(i-j)-1)\partial^2\Phi_i\partial^2\phi_j\Big).$$

- Using boson-fermion correspondence we will arrive to the Lagrangian which is supposed to describe the theory

$$\mathcal{L} = \sum_{k=1}^{n} \left(\frac{1}{8\pi}\left(\partial_\mu \Phi_k\right)^2 + i\bar{\psi}_k \gamma^\mu \partial_\mu \psi_k + \frac{\pi b^2}{2(1+b^2)}\left(\bar{\psi}_k \gamma^\mu \psi_k\right)^2 \right) -$$

$$- m\left(e^{b\Phi_1}\bar{\psi}_1\psi_1 + \sum_{k=2}^{n-1}\left(e^{b\Phi_k}\bar{\psi}_k\left(\frac{1+\gamma_5}{2}\right)\psi_k + \right.\right.$$

$$\left. + e^{-b\Phi_{k-1}}\bar{\psi}_k\left(\frac{1-\gamma_5}{2}\right)\psi_k \right) + e^{-b\Phi_{n-1}}\bar{\psi}_n\left(\frac{1-\gamma_5}{2}\right)\psi_n +$$

$$\left. + \cosh b\Phi_n \, \bar{\psi}_n \left(\frac{1+\gamma_5}{2}\right)\psi_n \right) -$$

$$- \frac{m^2}{8\pi b^2}\left(e^{2b\Phi_1} + 2\sum_{k=2}^{n-1} e^{b(\Phi_k - \Phi_{k-1})} + e^{b(\Phi_n - \Phi_{n-1})} + e^{-b(\Phi_{n-1}+\Phi_n)} \right).$$

- Last term corresponds to the counter term.

One can show that in the theories with fermionic screenings there are also screenings of sigma-model type. It means that for any two fermionic screenings with exponents $\boldsymbol{\alpha}_i$ and $\boldsymbol{\alpha}_j$ such that $(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j) \neq 0$ there is also the field

$$\mathcal{V}_{i,j} = (\boldsymbol{\alpha}_i, \partial\varphi)e^{(\boldsymbol{\beta}_{ij},\varphi)}, \quad \text{where} \quad \boldsymbol{\beta}_{ij} = \frac{2}{(\boldsymbol{\alpha}_i + \boldsymbol{\alpha}_j)^2}(\boldsymbol{\alpha}_i + \boldsymbol{\alpha}_j),$$

such that

$$\oint_{\mathcal{C}_z} W_{2k}(z)\mathcal{V}_{i,j}(w)dw = \partial_z \mathcal{V}_k^{(i,j)}(z).$$

This fact suggests the idea to consider the theory

$$\mathcal{L} = \frac{1}{8\pi}\Big(\partial_\mu\varphi\Big)^2 + \mu \sum_{(i,j)\in\mathbf{I}} (\boldsymbol{\alpha}_i, \partial\varphi)(\boldsymbol{\alpha}_i, \bar{\partial}\varphi)e^{(\boldsymbol{\beta}_{ij},\varphi)} + \ldots,$$

which might be the dual sigma-model description of the original theory.

By $\ldots$ we denoted possible counterterms. In order to take them into account we will use RG approach.

- Within this approach we have to study the one-loop evolution equation

$$R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Psi = -\dot{G}_{\mu\nu},$$

Namely, we are looking for the solution to this equation with the UV asymptotic prescribed by the bare Lagrangian

$$\mathcal{L} = \frac{1}{8\pi}\Big(\partial_\mu \varphi\Big)^2 + \mu \sum_{(i,j)\in \mathbf{I}} (\boldsymbol{\alpha}_i, \partial\varphi)(\boldsymbol{\alpha}_i, \bar{\partial}\varphi) e^{(\boldsymbol{\beta}_{ij},\varphi)} + \ldots,$$

- Here Ψ describes the effect of t-dependent diffeomorphisms (not all).

Consider the simplest case: $N = 5$. Let, (x_1, x_2, x_3, x_4) be the local coordinates in $\mathbb{R}^4$. We expect that the solution should behave in UV $t \to -\infty$ as

$$G_{\mu\nu} = \delta_{\mu\nu} + e^{\alpha t}\left(A_{\mu\nu}e^{x_1} + B_{\mu\nu}e^{-x_1-x_2} + C_{\mu\nu}e^{-x_1+x_2}\right) + \ldots,$$
$$\Psi = (\rho, x) + \ldots,$$

where

$$A_{\mu\nu} = \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & -1 \end{pmatrix}, \quad C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & -1 \end{pmatrix},$$

and ρ, some unknown constant vector.

At leading order we have

$$\alpha = \frac{3}{4}, \qquad \rho = \left(-\frac{1}{4}, 0, -\frac{3i}{4}, \frac{i}{2}\right).$$

Solving the asymptotic problem one can see that

$$G_{\mu\nu} = \begin{pmatrix} F_1 & 0 & iF_5 & 0 \\ 0 & F_2 - \cosh(x_2)F_6 & 0 & -i\sinh(x_2)F_6 \\ iF_5 & 0 & F_3 & 0 \\ 0 & -i\sinh(x_2)F_6 & 0 & F_4 + \cosh(x_2)F_6 \end{pmatrix}, \qquad \Psi = (\rho, x) + F_7.$$

These equations are compatible provided that

$$F_2 = F_4, \qquad F_2^2 = 1 + F_6^2.$$

The function F_7 is arbitrary. We can choose it such that the following relations holds

$$\det G = F_1 F_3 + F_5^2 = 1.$$

With this choice the problem has the unique solution

$$F_3 = \frac{(1-U)(1-UV)}{1-U^2V}, \qquad F_5 = \frac{U(1-V)}{1-U^2V} + \frac{2}{3}\frac{UV(1-U)}{1-U^2V},$$

$$F_6 = V^{\frac{1}{2}}\left(\frac{(1+V)}{(1-V)^2}\frac{1+U^2V}{1-U^2V} - \left(\frac{1}{2} + \frac{4V}{(1-V)^2}\right)\frac{U}{1-U^2V}\right),$$

$$F_7 = \log\left(\frac{(1-UV)^2}{1-V}\right)$$

where

$$U = e^{\frac{3t}{4}} e^{x_1}, \qquad V = \frac{1}{4} e^{\frac{3t}{2}} e^{-2x_1}.$$

It is convenient to introduce new coordinates ζ, θ, ϕ_1 and ϕ_2 by the following equations

$$F_3 = \frac{\kappa(1-\zeta^2)}{(1-\kappa^2\zeta^2)},\ \tanh\left(\frac{x_2}{2}\right) = \sin\theta,$$

$$\kappa = \frac{2-e^{\frac{3t}{2}}}{2+e^{\frac{3t}{2}}},\ \phi_1 = \frac{x_3}{2},\ \phi_2 = \frac{x_4}{2} - \frac{i}{2}\log\cos\theta.$$

In these coordinates the metric has the form (after rescaling $ds^2 \to 4\nu ds^2$, $t \to 4\nu t + \log 2$)

$$ds^2 = \frac{\kappa}{\nu}\left(\frac{d\zeta^2}{(1-\zeta^2)(1-\kappa^2\zeta^2)} + \frac{(1-\zeta^2)d\phi_1^2}{(1-\kappa^2\zeta^2)} + \zeta^2 d\theta^2 + 2i\zeta^2\tan\theta d\theta d\phi_2 + \right.$$
$$\left. + \frac{(1-\kappa^2\zeta^4\sin^2\theta)d\phi_2^2}{\kappa^2\zeta^2\cos^2\theta}\right).$$

We can also perform the T-duality in the ϕ_2 isometry making the metric diagonal

$$d\tilde{s}^2 = \frac{\kappa}{\nu}\left[\frac{d\zeta^2}{(1-\zeta^2)(1-\kappa^2\zeta^2)} + \frac{(1-\zeta^2)d\phi_1^2}{(1-\kappa^2\zeta^2)} + \frac{\zeta^2}{1-\kappa^2\zeta^4\sin^2\theta}\left(d\theta^2 + \cos^2\theta\, d\phi_2^2\right)\right]$$

and generating the non-zero pure imaginary $B-$field

$$B = \frac{i\kappa^2\sin\theta\cos\theta\zeta^4}{\nu(1-\kappa^2\zeta^4\sin^2\theta)} d\theta \wedge d\phi_2.$$

It can be checked that this metric and $B-$field coincide with the ones obtained from the $\eta = i\kappa$-deformed $O(5)$ sigma-model action mentioned in the introduction.

Conclusions

- Presumably, the duality should happen for all $\eta-$deformed SSSM. It is not known how to arrive to the system of screening fields starting from the deformed sigma-model action.

- At the moment it is unclear how to consider supergroup valued sigma-models in this context (important for AdS/CFT sigma-models).

- Once the dual theory is constructed. One has a system of local IM's. It is interesting to consider spectral problem for this system.

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