Cubic interactions of massless bosons in 3d

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Outline

- Introduction
- Cubic vertices for massless fields in any dimensions
- Cubic vertices in $d = 3$
- Implications for 2d CFT
- Conclusions

Introduction

- Massless higher-spin (HS) fields correspond to particles in $d \geq 4$, that are (almost) incompatible with non-trivial S-matrix in flat space. Not relevant for $d = 3$.
- Vasiliev equations in $(A)dS_d$ ($d \geq 3$) describe interacting theory of infinite tower of massless fields with spins $s = 0, 1, 2, 3, \ldots$, while action formulation is not known.
- Free action is available and cubic vertices are classified in any $d \geq 4$ (first step of the so-called Fronsdal program).
- In $d = 3$, an action formulation exists for both AdS and flat HS gravity, *without matter* (Chern-Simons). Matter-coupled Prokushkin-Vasiliev eq.'s lack action formulation though.
- Fronsdal program in $d = 3$ provides a way to couple matter to HS fields and construct action for Prokushkin-Vasiliev theory.

Formalism

Spin-s massless field is described by a rank-s symmetric tensor

$$\varphi^{(s)}(a,x) = \tfrac{1}{s!}\varphi_{\mu_1\dots\mu_s} a^{\mu_1}\cdots a^{\mu_s}\,,$$

where a^μ is an auxiliary vector, useful for handling arbitrary spin tensors. For example, divergence and trace are given as:

$$(\partial\cdot\partial_a)\varphi^{(s)}(a,x) = \tfrac{1}{(s-1)!}\partial^\nu\varphi_{\nu\mu_2\dots\mu_s} a^{\mu_2}\cdots a^{\mu_s}\,,$$
$$\Box_a\varphi^{(s)}(a,x) = \tfrac{1}{(s-2)!}\varphi^\nu{}_{\nu\mu_3\dots\mu_s} a^{\mu_3}\cdots a^{\mu_s}\,.$$

Here, $\partial_\mu = \frac{\partial}{\partial x^\mu}\,,\ \partial^a_\mu = \frac{\partial}{\partial a^\mu}$.

Free theory

Free action is given as

$$S = \frac{1}{2} \int d^d x \, \varphi^{(s)\mu_1 \dots \mu_s} \Box \varphi^{(s)}_{\mu_1 \dots \mu_s} + \dots \, ,$$

and free equations of motion are given by Fronsdal tensor, $\mathcal{F}^{(s)} = 0$

$$\mathcal{F}^{(s)}_{\mu_1 \dots \mu_s} = \Box \varphi^{(s)}_{\mu_1 \dots \mu_s} + \dots \, ,$$

linearised Einstein-Hilbert (Fierz-Pauli) action for $s = 2$.
There is a gauge symmetry with transformation law:

$$\delta \varphi^{(s)}_{\mu_1 \dots \mu_s} = \partial_{(\mu_1} \epsilon^{(s-1)}_{\mu_2 \dots \mu_s)} \, , \quad \epsilon^{(s-1)}(x, a) = \tfrac{1}{(s-1)!} \epsilon_{\mu_1 \dots \mu_{s-1}} a^{\mu_1} \cdots a^{\mu_{s-1}} \, .$$

Constraints: $\Box_a \epsilon^{(s-1)}(a, x) = 0 \, , \; \Box_a^2 \varphi^{(s)}(a, x) = 0.$

Interactions via Noether Method

Expand the action and gauge transformations in powers of fields:

$$S[\varphi] = S_2[\varphi] + g\, S_3[\varphi] + g^2\, S_4[\varphi] + \dots$$
$$\delta\varphi = \delta^0\varphi + g\,\delta^1\varphi + g^2\,\delta^2\varphi + \dots$$

The full action is gauge invariant: $\delta S[\varphi] = 0$, which implies:

$$\begin{aligned} \delta^0 S_2[\varphi] &= 0\,, \\ g(\delta^1 S_2[\varphi] + \delta^0 S_3[\varphi]) &= 0\,, \\ g^2(\delta^2 S_2[\varphi] + \delta^1 S_3[\varphi] + \delta^0 S_4[\varphi]) &= 0\,, \\ &\dots \end{aligned}$$

The solution to the first equation, i.e. the free action $S_2[\varphi]$ and $\delta^0\varphi$, are known.

Noether method at work

Cubic order in the action, $S_3[\varphi]$, satisfies the equation

$$\delta^1 S_2[\varphi] + \delta^0 S_3[\varphi] = 0$$

Taking into account, that $\delta S_2 \approx 0$ (for any $\delta\varphi$), where $\approx$ means equivalence up to free equations of motion, we are led to

$$\delta^0 S_3[\varphi] \approx 0$$

The cubic order Lagrangian can be written in the form

$$\mathcal{L}^{(3)} = \sum_{\{s_i\},n} g^n_{s_1,s_2,s_3} \mathcal{L}^n_{s_1,s_2,s_3} \,,$$

n counts independent vertices for three fields with arbitrary spins:

$$\mathcal{L}^n_{s_1,s_2,s_3} = \mathcal{V}^n_{s_1,s_2,s_3}(\partial_{a_i}, \partial_i)\, \varphi^{(s_1)}(a_1, x_1)\varphi^{(s_2)}(a_2, x_2)\varphi^{(s_3)}(a_3, x_3)$$

Vertex operator algebra

A basis of building blocks for the vertex operators are given in our language (we discard total derivatives)

$$B_{ij} = \partial_i \cdot \partial_j \,, \quad y_i = \partial_{a_i} \cdot \partial_{i+1} \,, \quad z_i = \partial_{a_{i+1}} \cdot \partial_{a_{i-1}} \,,$$
$$Div_i = \partial_{a_i} \cdot \partial_i \,, \quad Tr_i \equiv \Box_{a_i} = \partial_{a_i} \cdot \partial_{a_i} \,.$$

- We restrict ourselves for simplicity to Traceless-Transverse part of the vertex, discarding Div_i and Tr_i operators.
- We discard total derivatives and fix field redefinition freedom uniquely so that there are no operators B_{ij} in the vertex.

Simplification

The traceless-transverse part of the cubic vertex for arbitrary spin massless bosons depends only on six structures y_i, z_i: $\mathcal{V}(y_i, z_i)$.

Cubic vertices in any $d \geq 4$

Using commutators $[y_i, a_j \cdot \partial_j] \approx 0\,,\ [z_{i\pm1}, a_i \cdot \partial_i] \approx \pm y_{i\mp1}\,$, the equation $\delta^0 \mathcal{L}^{(3)} \approx 0$ is reformulated as:

$$(y_{i-1}\partial_{z_{i+1}} - y_{i+1}\partial_{z_{i-1}})\mathcal{V}(y_i, z_i) \approx 0\,.$$

The solution is simple:

$$\mathcal{V}(y_i, z_i) = C(y_i, G)\,, \quad G = y_1 z_1 + y_2 z_2 + y_3 z_3$$

$\mathcal{V}^n_{s_1,s_2,s_3}$ vertex with $s_1 + s_2 + s_3 - 2n \geq max\{s_i\}$ derivatives

$$\mathcal{V}^n_{s_1,s_2,s_3}(y_i, z_i) = y_1^{s_1-n} y_2^{s_2-n} y_3^{s_3-n} G^n\,, \quad (0 \leq n \leq min\{s_i\})$$

One vertex for each number of derivatives in the allowed range. Altogether $min\{s_1, s_2, s_3\} + 1$ vertices. Matches the structures of three-point functions of conserved currents in CFT_{d-1}.

Cubic vertices in $d \geq 3$: examples

pure spin-one vertices:

$\mathcal{V}^0_{1,1,1} = y_1\, y_2\, y_3$ (F^3 vertex).

$\mathcal{V}^1_{1,1,1} = G$ (Yang-Mills vertex).

Pure spin-two vertices:

$\mathcal{V}^0_{2,2,2} = y_1^2\, y_2^2\, y_3^2$ (W^3 vertex).

$\mathcal{V}^1_{2,2,2} = y_1\, y_2\, y_3\, G$ (Gauss-Bonnet vertex).

$\mathcal{V}^2_{2,2,2} = G^2$ (Einstein-Hilbert vertex).

Cubic vertices in $d = 3$

In 3d, the construction gets modified by Schouten identities (SI), i.e. operators, contracting $\delta^{\nu_1\nu_2\nu_3\nu_4}_{\mu_1\mu_2\mu_3\mu_4}$ and $\partial^{\mu}_{a_i}$, ∂^{ν}_{i} operators.

$$(y_{i-1}\partial_{z_{i+1}} - y_{i+1}\partial_{z_{i-1}})\mathcal{V}(y_i, z_i) \approx SI \equiv 0\,.$$

The list of elementary Schouten identities, relevant to the cubic vertex problem is given as (no summation over i):

$$\begin{aligned} (G - y_i z_i)^2 &= 0\,, \qquad & y_i z_i G - y_{i-1} z_{i-1} y_{i+1} z_{i+1} &= 0\,, \\ y_i y_{i\pm1}(G - y_i z_i) &= 0\,, & & \\ y_i^2 y_{i+1}^2 &= 0\,, & y_i^2 y_{i+1} y_{i-1} &= 0\,. \end{aligned}$$

Due to these identities, the classification of three dimensional vertices is different from $d \geq 4$.

Cubic vertices in 3d: massless scalar and vector fields

For simplicity, we assume $s_1 \geq s_2 \geq s_3$ and derive the parity-even vertices only. All the vertices with scalar fields ($s_3 = 0$) are:

$$\mathcal{V}_{s,0,0} = y_1^s \,,$$
$$\mathcal{V}_{s,1,0} = y_1^s \, y_2 \,,$$

while all the vertices with Maxwell vector fields ($s_3 = 1$) are:

$$\mathcal{V}_{s,1,1} = y_1^{s-1} G \,,$$
$$\mathcal{V}_{s,s,1} = y_1 \, y_2 \, y_3 \, z_3^{s-1}$$

The only example of spin configurations in three dimensions, with more than one vertex, is $s_1 = s_2 = s_3 = 1$:

$$\mathcal{V}^{YM}_{1,1,1} = G \,, \quad \mathcal{V}^{F^3}_{1,1,1} = y_1 \, y_2 \, y_3 \,.$$

Cubic vertices in 3d: gravitational coupling

Massless spin two interactions ($s_3 = 2$) are interesting as in three dimensions one can have minimal coupling to gravity:

$$\mathcal{V}_{s,s,2} = y_3\, z_3^{s-1}(s\, y_1 z_1 + s\, y_2 z_2 + y_3 z_3)\,.$$

This vertex could be also derived by simply covariantising derivatives in massless spin$-s$ free action.
It does not spoil gauge invariance of spin$-s$ field only in three dimensions due to triviality of Weyl tensor, which implies that the Riemann tensor is algebraically related to Ricci tensor - equation of motion for metric.
This naturally triggers the possibility to have a minimal coupling to gravity, at the expense of deforming the gauge transformation of the metric and enlarging the space-time symmetries exactly in the same way as introducing massless spin $\frac{3}{2}$ field leads to SUGRA.

Cubic vertices in 3d: general triples $s_1 \geq s_2 \geq s_3 \geq 2$

No vertices for $s_1 \geq s_2 + s_3$.

For $s_1 < s_2 + s_3$, we have two cases:

$s_1 + s_2 + s_3$ is even. There is a unique two-derivative vertex:

$$\mathcal{V}_{s_1,s_2,s_3} = [(s_1 - 1)y_1 z_1 + (s_2 - 1)y_2 z_2 + (s_3 - 1)y_3 z_3] G z_1^{n_1} z_2^{n_2} z_3^{n_3}, \quad n_i = \tfrac{1}{2}(s_{i-1} + s_{i+1} - s_i) - 1 \geq 0.$$

$s_1 + s_2 + s_3$ is odd. There is a unique three-derivative vertex:

$$\mathcal{V}_{s_1,s_2,s_3} = y_1\, y_2\, y_3\, z_1^{n_1} z_2^{n_2} z_3^{n_3}\,, \quad n_i = \tfrac{1}{2}(s_{i-1} + s_{i+1} - s_i - 1) \geq 0\,.$$

Cubic vertices in 3d: parity-odd case for $s_1 \geq s_2 \geq s_3 \geq 2$

No vertices for $s_1 \geq s_2 + s_3$.

For $s_1 < s_2 + s_3$, we have two cases:

$s_1 + s_2 + s_3$ is odd. There is a unique two-derivative vertex:

$$\mathcal{V}^{PO}_{s_1,s_2,s_3} = [n_1\, W_1 z_1 + n_2\, W_2 z_2 + n_3\, W_3 z_3]\, z_1^{n_1} z_2^{n_2} z_3^{n_3}\,,$$
$$W_i = \epsilon^{\mu\nu\lambda} \partial^{a_i}_\mu \partial^{i+1}_\nu \partial^{i-1}_\lambda\,, \quad n_i = \tfrac{1}{2}(s_{i-1} + s_{i+1} - s_i - 1) \geq 0\,.$$

$s_1 + s_2 + s_3$ is even. There is a unique three-derivative vertex:

$$\mathcal{V}_{s_1,s_2,s_3} = y_1\, y_2\, y_3\, U\, z_1^{n_1} z_2^{n_2} z_3^{n_3}\,,$$
$$U = \epsilon^{\mu\nu\lambda} \partial^{a_1}_\mu \partial^{a_2}_\nu \partial^{a_3}_\lambda\,, \quad n_i = \tfrac{1}{2}(s_{i-1} + s_{i+1} - s_i) - 1 \geq 0\,.$$

Cubic vertices in 3d: parity-odd vs parity-even

Key observation:

$$U^2 = -2\, z_1\, z_2\, z_3 \;\rightarrow\; z_1^{1/2}\, z_2^{1/2}\, z_3^{1/2} = \frac{i}{\sqrt{2}} U$$

In the two-derivative parity-odd vertex perform the substitution $n_i \rightarrow n_i + \frac{1}{2}$ (and use that $W_i\, U = y_i(G - y_i\, z_i)$)

$$\mathcal{V}^{PO}_{s_1,s_2,s_3} \rightarrow [(n_1 + \tfrac{1}{2})\, W_1 z_1 + (n_2 + \tfrac{1}{2})\, W_2 z_2 + (n_3 + \tfrac{1}{2})\, W_3 z_3] \times \frac{i}{\sqrt{2}} U\, z_1^{n_1}\, z_2^{n_2}\, z_3^{n_3}$$

$$= \frac{i}{\sqrt{2}} [s_1 y_1 z_1 + s_2 y_2 z_2 + s_3 y_3 z_3]\, G\, z_1^{n_1}\, z_2^{n_2}\, z_3^{n_3} = \mathcal{V}_{s_1+1, s_2+1, s_3+1}\,.$$

Conclusion:

Parity-even and parity-odd vertices are related by half-integer shift of exponents of z_i or $s_i \rightarrow s_i + 1$, and the above replacement.

Cubic vertices in 3d: Chern-Simons couplings

Minimal coupling to CS vector field for any spin$-s$ field

Direct computation shows that there is a minimal coupling $(s, s, 1)$ with one derivative, given as:

$$\mathcal{V}_{\mathcal{CS}} = y_3 \, z_3^s$$

Alternative derivation:

Let us take the free action of a spin$-s$ field and replace derivatives with covariant ones $\partial_\mu \to \nabla_\mu = \partial_\mu + A_\mu$. The obstruction terms for spin$-s$ gauge invariance are related to commutators $[\nabla_\mu, \nabla_\nu] = F_{\mu\nu}$. These are CS equations of motion!

Conclusion:

CS minimal coupling works exactly same way as minimal coupling to gravity. CS field gets deformation of gauge transformation via gauge parameter of the spin$-s$ field.

More vertices with CS fields. See arXiv:1803.02737 for the full list.

Cubic vertices in 3d: beyond Minkowski

This classification extends to arbitrary Einstein background.

In full theory, Einstein eq.'s get corrections quadratic in HS fields. Einstein spaces are still solutions of the whole system, with zero background values of HS fields. We can expand around these solutions, and the cubic vertices can be extended straightforwardly.

Know how.

One can simply replace $\partial \to \nabla$, to get the vertex in arbitrary Einstein spaces, treating gravity in full non-linear manner, while still expanding in powers of HS (weak) fields.

Cubic vertices in AdS_3 vs 3pt functions in CFT_2

Match to CFT

For $s_1 \geq s_2 \geq s_3 \geq 2$ and $s_1 < s_2 + s_3$ we have one-to-one match of the number of cubic vertices in AdS_3 and independent structures in 3pt functions of conserved currents on the boundary.
There are two vertices for every triple — one parity-even and one parity-odd.
There are two independent structures in three-point functions — chiral and anti-chiral.

Puzzle

The absence of the vertices for $s_1 \geq s_2 + s_3$ is puzzling as there are structures of 3pt functions in CFT_2, corresponding to such vertices.

Conclusions

Main Points

- Exhaustive classification of cubic vertices for massless bosons in three dimensions. Minimal coupling to gravity and Chern-Simons vector fields as particular examples.
- As opposed to $d \geq 4$, there is at most one parity-even and one parity-odd vertex for each triple of spins (with one exception),
- They can be extended to arbitrary Einstein background, and match CFT 3pt functions wherever applicable.
- Relation between parity-even and parity-odd vertices.
- A non-trivial model-independent property of any CFT_2 with HS conserved currents uncovered.
- This is the first step towards Lagrangian theories of HS fields with matter in $d = 3$.

Additional slides follow

Free theory

Free action is given as

$$S=\frac{1}{2}\int d^dx\,\varphi^{(s)}(a,x)\star\Big[1-\frac{1}{4}\,a^2\,\Box_a\Big]\mathcal{F}^{(s)}(a,x)\,,$$

and free equations of motion are given by Fronsdal tensor, $\mathcal{F}^{(s)}=0$

$$\mathcal{F}^{(s)}(a,x)=\Big[\Box-(a\cdot\partial_x)(\partial_a\cdot\partial_x)+\frac{1}{2}(a\cdot\partial)^2\Box_a\Big]\varphi^{(s)}(a,x)\,,$$

linearised Einstein-Hilbert (Fierz-Pauli) action for $s=2$.
There is a gauge symmetry with transformation law:

$$\delta\varphi^{(s)}(a,x)=(a\cdot\partial)\epsilon^{(s-1)}(a,x)\,,\ \epsilon^{(s-1)}(x,a)=\tfrac{1}{(s-1)!}\epsilon_{\mu_1\dots\mu_{s-1}}a^{\mu_1}\cdots a^{\mu_{s-1}}$$

Constraints: $\Box_a\epsilon^{(s-1)}(a,x)=0\,,\ \Box_a^2\varphi^{(s)}(a,x)=0.$