Spinning Mellin Bootstrap

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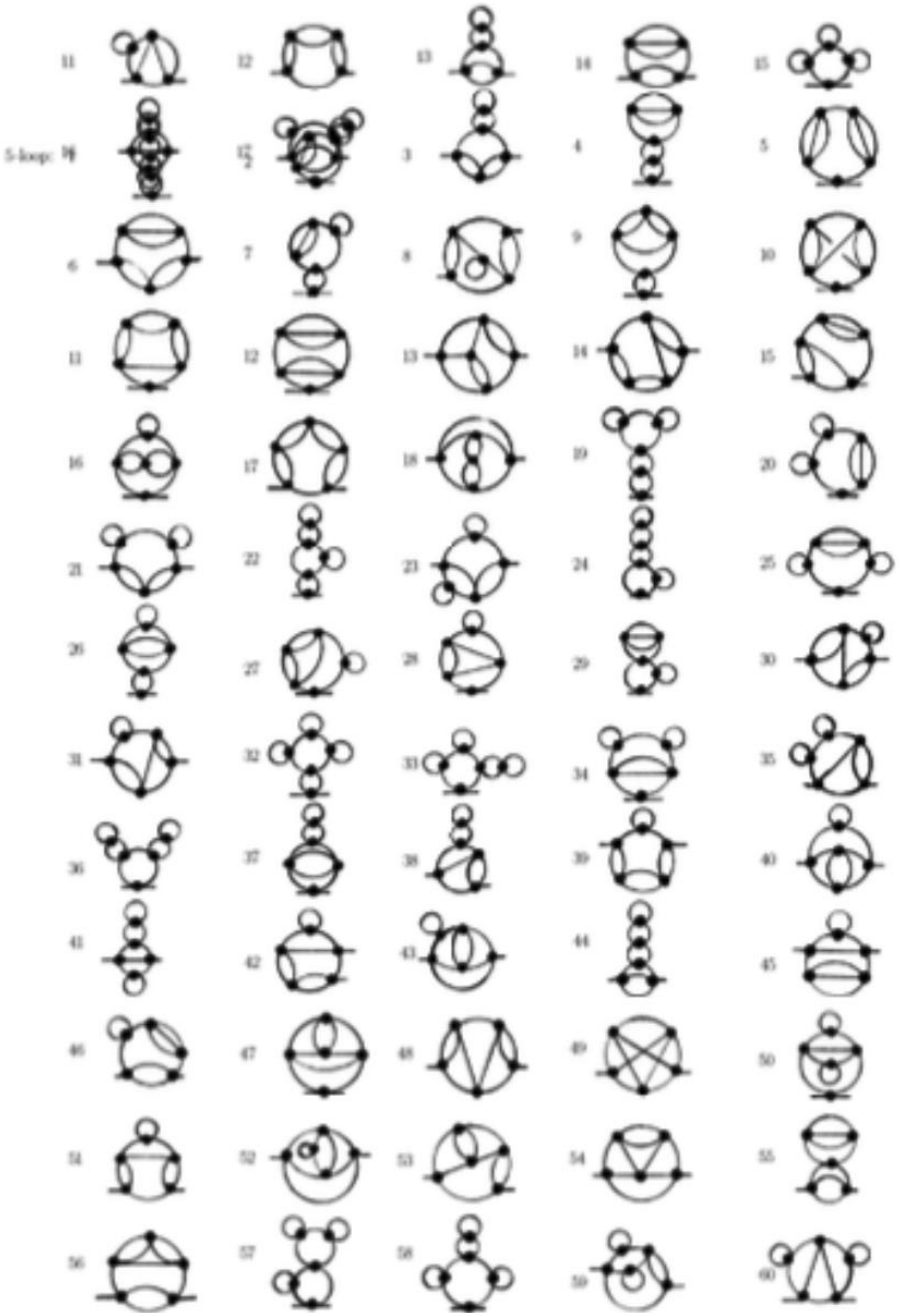
Mostly based on: 1702.08619, 1708.08404, 1804.09334 & to appear

w. Charlotte Sleight

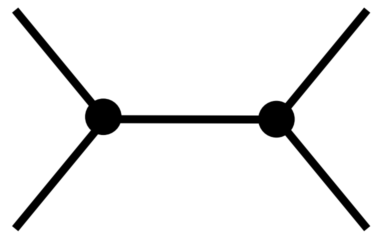
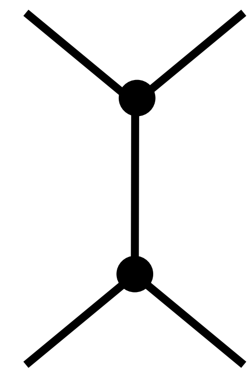


Feynman diagrams quickly become unmanageable, however final result of the resummation of many complicated diagram is very often simple

Use symmetry & Quantum Mechanics to find the answer directly

Challenges in Analytic Bootstrap

The basic idea is to bypass Feynman diagram (bulk or boundary) computation and just impose:

$\sum_{l,\Delta} c_{l,\Delta}$  $= \sum_{l,\Delta} c_{l,\Delta}$ 

Goals (if time permits):

- use Mellin space to uncover explicitly inversion formulas!
- Clarify the role of AdS/CFT at tree level (kinematic transform)
- Demystify “holographic reconstruction”: equivalence between Noether procedure and bootstrap at tree-level
- Revisit the role of differential operators to generate spinning blocks in terms of higher-spin generators
- …

Inversion formulas in S-matrix

Inversion formula are standard tools in S-matrix theory

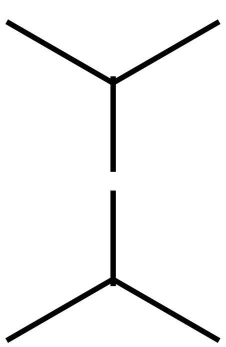
$$\mathcal{M}(s,t(\theta)) = \sum_J a_J(s)\, C_J(\cos\theta)$$

Partial waves
(fixed by isometries – kinematics)

$$a_J(s) = \int_{-1}^{1} d(\cos\theta)(\sin\theta)^{d-4} C_J(\cos\theta)\, \mathcal{M}(s,t(\theta))$$

Obtain the spin J coefficient directly from S-matrix

Study the above problem as a function of spin: “continuous spin”



$$= \sum_l c_J(s) C_J(\cos\theta)$$

Inversion formulas & Bulk locality

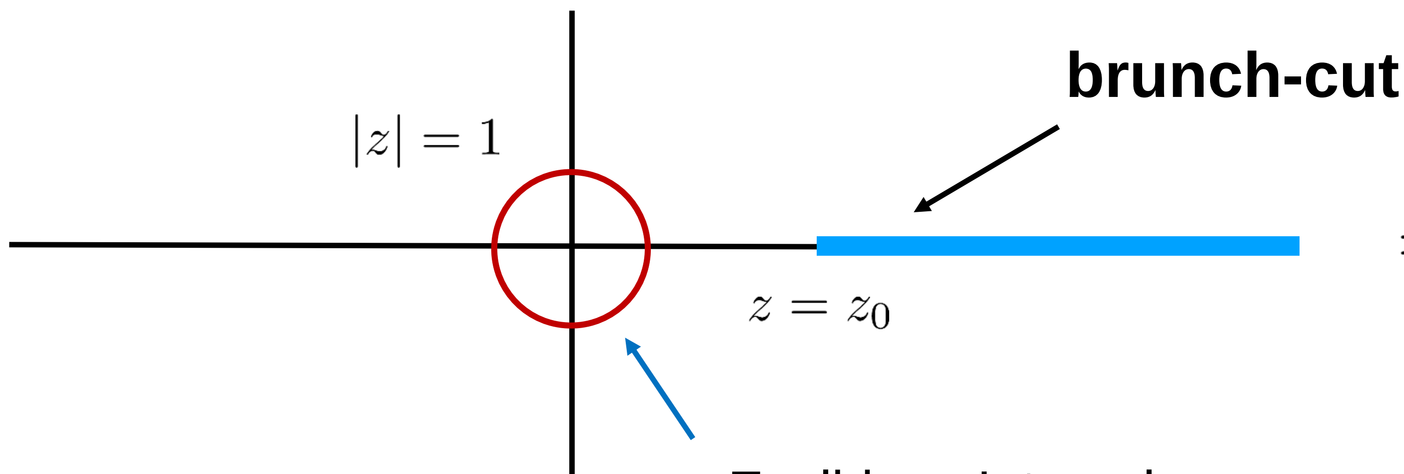
What is the interpretation of analyticity in spin?

$$f(z) = \sum_l f_l \, z^l \qquad\qquad \lim_{z\to 0}\left|\frac{f(z)}{z}\right| = 0$$

Inversion = Cauchy

$$f_j = \oint \frac{dz}{z} z^{-l} f(z)$$

$f_l =$



$$= \frac{1}{2\pi i}\int_{z_0}^{\infty} \frac{dz}{z} z^{-l}\, \mathrm{Disc} f(z)$$

Euclidean Integral
around origin (l integer)

analytic for continuous *l*:
continuous spin

$$l > 1$$

Coefficients form an infinite family and have to be varied analytically

Inversion formulas in CFT

Inversion formula are standard tools in Harmonic analysis: diagonalize Casimir in a way that preserves self-adjointness

$$\mathcal{C}_2 = \frac{1}{2} L_{AB} L^{AB}$$

$$\langle f | g \rangle \sim \int du\, dv\, \mu(u,v)\, f(u,v)\, g(u,v) \qquad \langle f | \mathcal{C}_2\, g \rangle = \langle \mathcal{C}_2 f | g \rangle$$

$$u = \frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2} \qquad v = \frac{y_{14}^2 y_{23}^2}{y_{13}^2 y_{24}^2}$$

Self-adjointness requires that f and g are single valued functions (in Euclidean kinematics)

An orthogonal basis of eigenfunctions of the Casimir can be found in terms of conformal partial waves

Conformal Partial Wave (CPW)

$$F_{l',\Delta} = G_{J,\Delta}(u,v) - \#\, G_{J,d-\Delta}(u,v) \sim u^{\frac{\Delta-J}{2}} [g(v) + O(u)] - \# u^{\frac{d-\Delta-J}{2}} [\tilde{g}(v) + O(u)]$$

Conformal block **Shadow**

$$\boxed{\langle \phi\phi\phi\phi \rangle = \# \int_{-i\infty}^{+i\infty} \frac{d\Delta}{2\pi i} \sum_l c_l(\Delta)\, F_{l,\Delta}(u,v) + \underset{\Delta \leq \frac{d}{2}}{\text{non-normalisable}}}$$

Inversion formulas

Inversion formula are standard tools in Harmonic analysis: diagonalize Casimir in a way that preserves self-adjointness

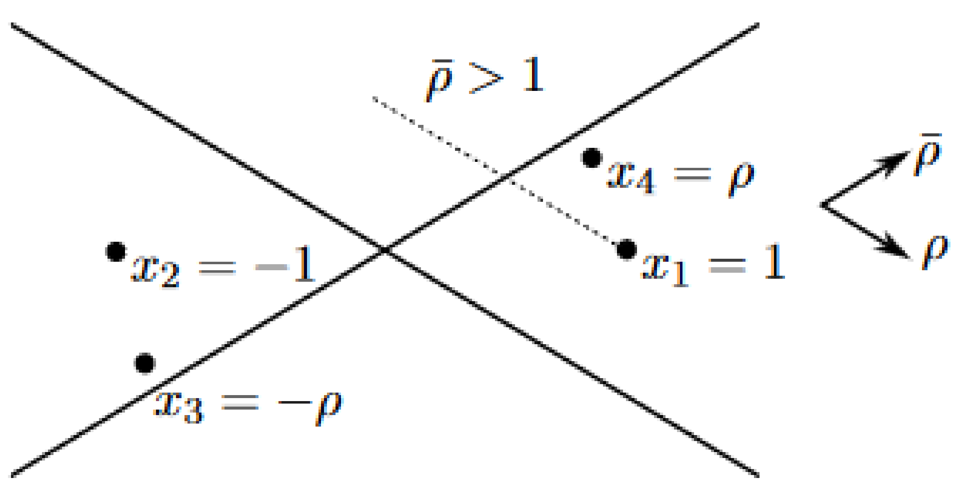
$$c(l', \Delta) = \# \int du\, dv\, \mu(u,v)\, F_{l',\Delta}(u,v)\ \langle \phi\phi\phi\phi \rangle$$

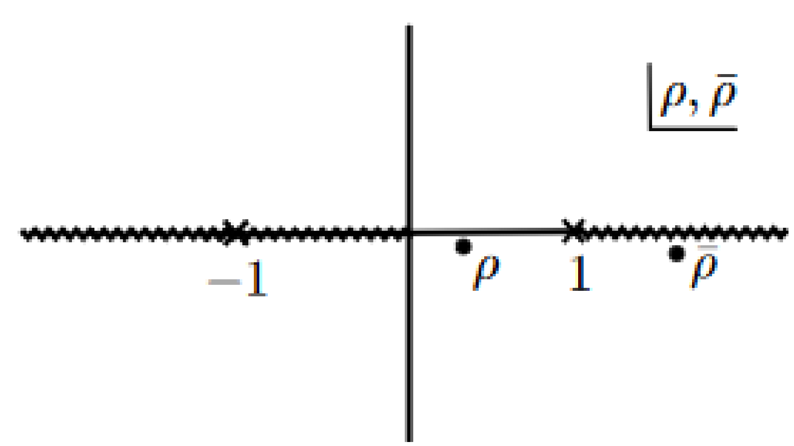
Ortogonality & completeness requires that Delta is on the principal series: $\Delta = \frac{d}{2} - i\nu$

$\sim\ e^{-i\,E\,x_0}$

The above Euclidean formula is the basis for recent developments of analytic bootstrap [Alday et al., Caron-Huot, …]

$x_4 = -x_3 = (\rho, \bar{\rho})$ $\qquad$ $x_1 = -x_2 = (1,1)$





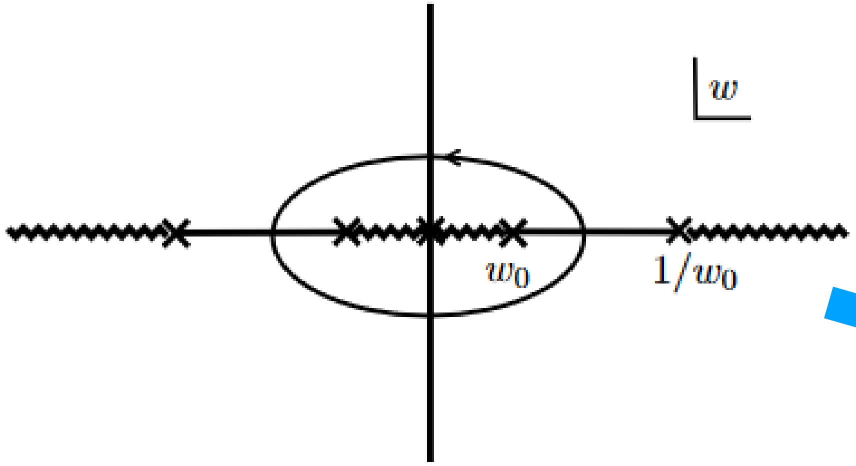
Regge Limit: $\rho = \dfrac{\sigma}{E}$ $\qquad$ $\bar{\rho} = E$ $\qquad$ $E \to \infty$ $\qquad$ $\sigma = \text{const.}$

Inversion formulas

Toy example of analytic continuation in S-matrix theory (d=2)

$$c(l,s) = \# \oint \frac{dw}{w} \, w^l \mathcal{M}(s,t(w)) \qquad w = e^{i\theta}$$

Gegenbauer polynomials in d=2



Close the contour to the center if J big enough to obtain an integral over a discontinuity

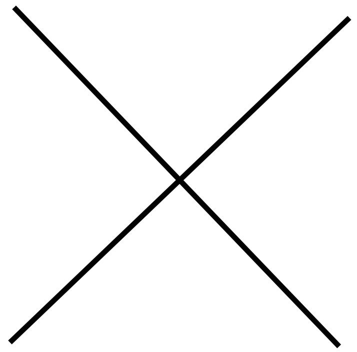
Similar steps in CFT allow to express the function c in terms of the double-discontinuity of the correlator

$$c(l,\Delta) = \int_0^1 dudv \, G_{\Delta+1-d,l+d-1}(u,v) \, \mathrm{dDisc}[\langle\phi\phi\phi\phi\rangle]$$

The CPW gets analytically continued into a conformal block with spin and dimension interchanged (analyticity in spin/continuous spin)

[Caron-Huot, 2017]

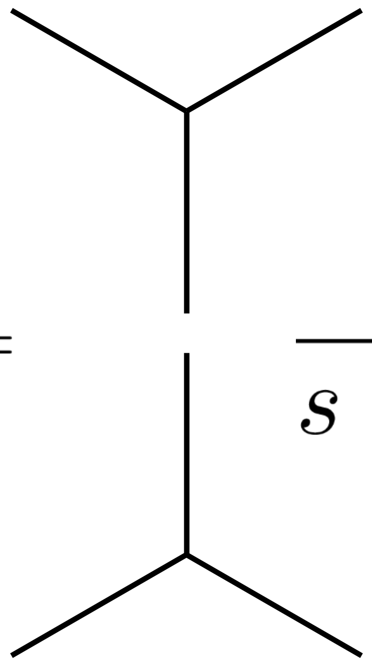
Inversion formulas & Bulk locality



$$\sim \frac{f_0}{\Delta^2} + \frac{f_1\, s}{\Delta^4} + \dots$$

Inversion formula tells us that the only free parameters are the first 2

$$\sum \left(\frac{\Box}{\Delta^2}\right)^n \to \quad \frac{\color{red}{f_0}}{\Delta^2} + \frac{\color{red}{f_1}\, s}{\Delta^4} + \frac{f_2\, s^2}{\Delta^6} + \dots =$$



$$\frac{1}{s - \Delta^2}$$

All other terms have to resum to reproduce the discontinuity of the amplitude (EFT).

Contact terms beyond the first few must resum to 1/Box (EFT)

[Sleight & M.T. 2017]

Mellin space

So far all integral formulas we wrote required careful analysis of the conformal integrals involved (gauge fixing etc…)

is there a way to make manifest these orthogonality properties?

$$\int d^d y' \frac{1}{[(y-y')^2]^{d-\Delta}} \mathcal{O}_\Delta(y')$$

$$F_{l=0,\Delta} = \# \int d^d y_0 \; \langle\langle \mathcal{O}_{\Delta_1}(y_1)\mathcal{O}_{\Delta_2}(y_2)\mathcal{O}_{\Delta,0}(y_0)\rangle\rangle \; \langle\langle \tilde{\mathcal{O}}_{\Delta,0}(y_0)\mathcal{O}_{\Delta_3}(y_3)\mathcal{O}_{\Delta_4}(y_4)\rangle\rangle$$

$$\sim \# \frac{1}{(y_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta}{2}}(y_{34}^2)^{\frac{\Delta_3+\Delta_4-(d-\Delta)}{2}}} \int d^d y_0 \underbrace{\frac{1}{(y_{01}^2)^{\frac{\Delta+\Delta_1-\Delta_2}{2}}(y_{20}^2)^{\frac{\Delta_2+\Delta-\Delta_1}{2}}(y_{03}^2)^{\frac{(d-\Delta)+\Delta_3-\Delta_4}{2}}(y_{40}^2)^{\frac{\Delta_4+(d-\Delta)-\Delta_3}{2}}}}_{\text{Standard 4pt conformal integral}}$$

$$\boxed{\sim \# F_{l,\Delta}(u,v)}$$

Symanzik star formula allows to evaluate these integral in terms of a Mellin representation

$$F_{l,\Delta}(u,v) = \# \int \frac{ds\, dt}{(4\pi i)^2} \, u^{t/2} v^{-(s+t)/2} \rho(s,t) \; F_{\Delta,l}(s,t)$$

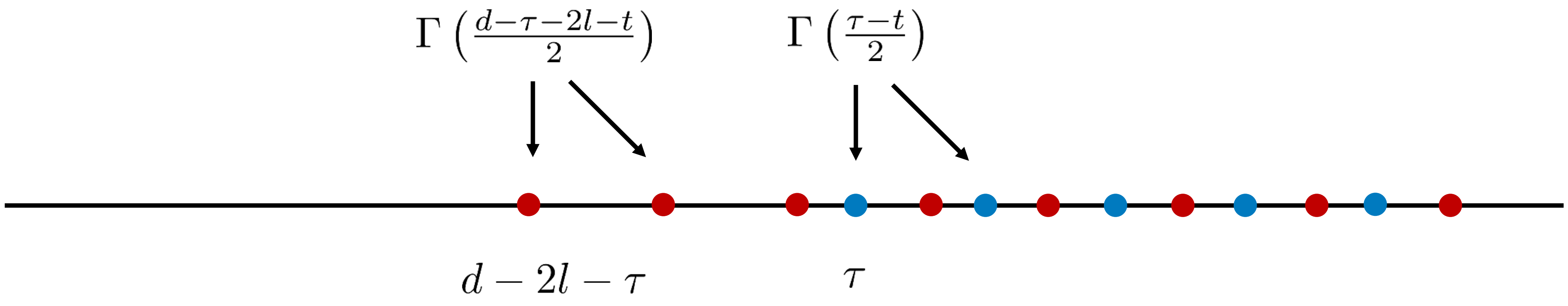
Mack Polynomials

$$\rho(s,t) = \Gamma\left(\tfrac{-t+2\Delta}{2}\right)^2 \Gamma\left(\tfrac{s+t}{2}\right)^2 \Gamma\left(-\tfrac{s}{2}\right)^2$$

$$F_{\Delta,l}(s,t) \sim \frac{\Gamma\left(\frac{\tau-t}{2}\right)\Gamma\left(\frac{d-\tau-2l-t}{2}\right)}{\Gamma\left(\frac{-t+2\Delta}{2}\right)} P_{l,\Delta}(s,t)$$

Mellin space

Mack polynomials encode conformal partial waves in terms of degree *l* polynomials in analogues of Mandelstam variables



For each primary operator we have an infinite series of poles:

$$t = \tau + 2m \quad \begin{cases} m = 0 & \text{Physical pole} \\ m > 0 & \text{Descendants pole} \end{cases}$$

Projecting out the shadow poles is straightforward: [Fitzpatrick & Kaplan 2011]

$$G_{l,\tau}(s,t) \sim \left(e^{i\pi(t+\tau+2l-d)} - 1\right) P_{l,\tau+l}(s,t)$$

Mellin space

Orthogonality of CPWs becomes manifest in Mellin space: [Costa et al.]

$$P_{l,\tau}(s,t) \sim \sum_m \frac{\mathcal{Q}_{l,m}(s)}{t-\tau-2m} \quad \left\{ \begin{array}{l} \rho(s,t) \to \Gamma\left(\frac{s+\tau}{2}\right)^2 \Gamma\left(-\frac{s}{2}\right)^2 \\ \\ \mathcal{Q}_{l,0} \sim \mathfrak{N}^{-1}\, Q_l^{(\tau,\tau,0,0)}(s) \end{array} \right.$$

The kinematic polynomials turn out to be Continuous Hahn polynomials (3F2)

$$\langle f(s) g(s) \rangle_{a,b,c,d} = \int_{-i\infty}^{i\infty} \frac{ds}{4\pi i}\, \Gamma(\tfrac{s+a}{2})\Gamma(\tfrac{s+b}{2})\Gamma(\tfrac{c-s}{2})\Gamma(\tfrac{d-s}{2})\, f(s)\, g(s)$$

$$Q_l^{(a,b,c,d)}(s) = \frac{(-2)^l \left(\frac{a+c}{2}\right)_l \left(\frac{a+d}{2}\right)_l}{\left(\frac{a+b+c+d}{2}+l-1\right)_l}\, {}_3F_2\left(\begin{matrix} -l,\ \frac{a+b+c+d}{2}+l-1,\ \frac{a+s}{2} \\ \frac{a+c}{2},\ \frac{a+d}{2} \end{matrix};1\right) \sim s^l + \ldots$$

Position space orthogonality becomes manifest in Mellin space!

$$c(l,\Delta) \sim \int \frac{ds}{4\pi i}\, \rho(s,\tau)\, \mathcal{M}(s,\tau) Q_l^{(\tau,\tau,0,0)}(s)$$

What about spinning external legs?

Spinning Correlators

Spinning correlators require to introduce tensorial structures

$$\mathsf{Y}_{i,jk} = \frac{z_i \cdot y_{ij}}{y_{ij}^2} - \frac{z_i \cdot y_{ik}}{y_{ik}^2} \qquad \mathsf{H}_{ij} = \frac{1}{y_{ij}^2}\left(z_i \cdot z_j + \frac{2 z_i \cdot y_{ij} z_j \cdot y_{ji}}{y_{ij}^2}\right)$$

3pt functions can be decomposed in terms of monomials:

$$z_i \cdot z_i = 0$$

$$\langle\langle \mathcal{O}_{\Delta_1,J_1}(y_1)\mathcal{O}_{\Delta_2,J_2}(y_2)\mathcal{O}_{\Delta_3,J_3}(y_3)\rangle\rangle^{(\mathbf{n})} = \frac{\mathfrak{I}^{n_1,n_2,n_0}_{J_1,J_2,J_3}}{(y_{12}^2)^{\frac{\tau_1+\tau_2-\tau}{2}}(y_{23}^2)^{\frac{\tau_2+\tau-\tau_1}{2}}(y_{31}^2)^{\frac{\tau+\tau_1-\tau_2}{2}}}$$

$$\mathfrak{I}^{n_1,n_2,n_3}_{J_1,J_2,J_3} = \mathsf{Y}_{1,23}^{J_1-n_2-n_3}\mathsf{Y}_{2,31}^{J_2-n_3-n_1}\mathsf{Y}_{3,12}^{J_3-n_1-n_2}\mathsf{H}_{23}^{n_1}\mathsf{H}_{31}^{n_2}\mathsf{H}_{12}^{n_3}$$

Conformal symmetry allows to reconstruct the correlator from a subset of the structures

$$W_{ij} = \frac{z_i \cdot y_{ij}}{y_{ij}^2} \qquad \left\{ \begin{array}{l} \langle \mathcal{O}_{\Delta_1,J_1}(y_1)\mathcal{O}_{\Delta_2,J_2}(y_2)\mathcal{O}_{\Delta_3,J_3}(y_3)\rangle \sim \# f^{(3)}(W_{ij}) + \mathcal{O}(z_i \cdot z_j) \\ \langle \mathcal{O}_{\Delta_1,J_1}(y_1)\mathcal{O}_{\Delta_2,J_2}(y_2)\mathcal{O}_{\Delta_3,J_3}(y_3)\mathcal{O}_{\Delta_4,J_4}(y_4)\rangle \sim \# f^{(4)}(W_{ij}) + \mathcal{O}(z_i \cdot z_j) \end{array} \right.$$

Spinning CPWs

The definition of CPWs given in the scalar case is very general

$$F^{\mathbf{n},\bar{\mathbf{n}}}_{\tau,l}(y_i) \sim \int d^d y_0 \, \langle\langle \mathcal{O}_{\Delta_1,J_1}(y_1)\mathcal{O}_{\Delta_2,J_2}(y_2)\mathcal{O}_{\Delta,l}(y_0)\rangle\rangle^{(\mathbf{n})} \, \langle\langle \tilde{\mathcal{O}}_{\Delta,l}(y_0)\mathcal{O}_{\Delta_3,J_3}(y_3)\mathcal{O}_{\Delta_4,J_4}(y_4)\rangle\rangle^{(\bar{\mathbf{n}})}$$

$$\Downarrow$$

$$F^{\mathbf{n},\bar{\mathbf{n}}}_{\tau,l}(s,t|W_{ij})$$

The above integral can be explicitly performed in Mellin space but without a guiding principle its form does not show any structure

Orthogonality is not manifest because it involves a delicate interplay between different tensor structures…

We will argue that a key guiding principle lies in the bulk-to-boundary map: (AdS/CFT)

Tree level AdS/CFT ~ momentum space

$\mathbf{AdS_{d+1}}$ | $\mathbf{CFT_d}$

Fields
Single-particle state

$$\varphi_s$$

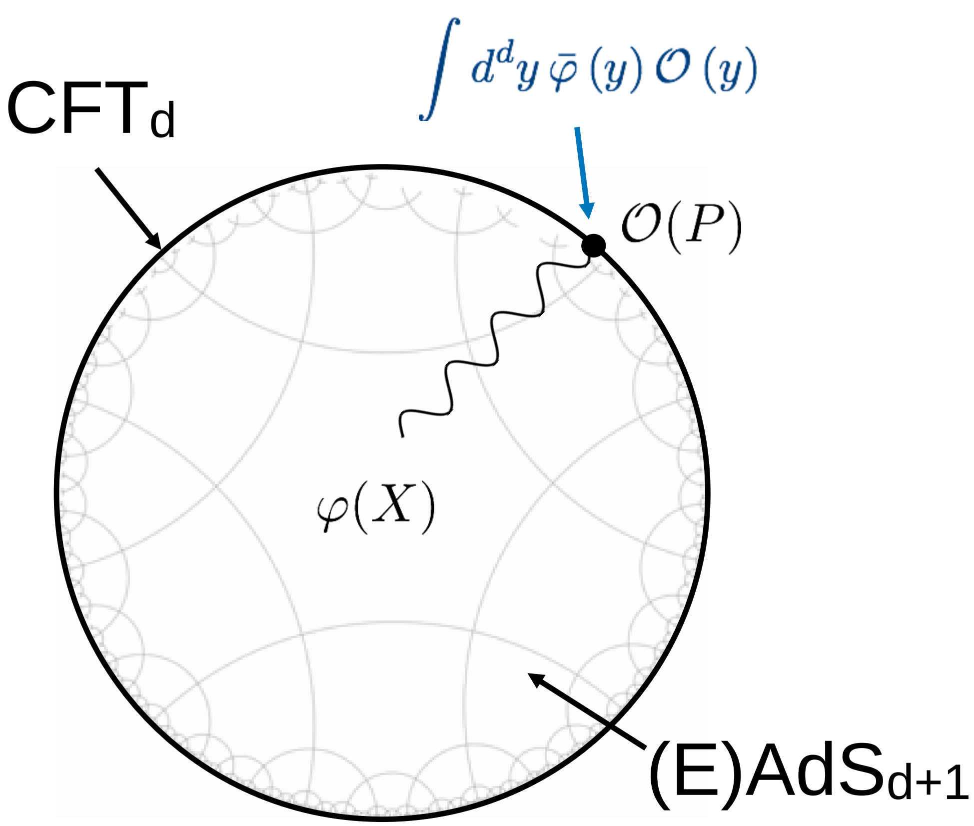
$$(mR)^2 = \Delta\,(\Delta - d) - s$$

$\longleftrightarrow$

Single-trace
Primary operator

$$\mathcal{O}_{\Delta,s}$$

scaling dim. spin



$$\left\{ \begin{aligned} &(\nabla^2 + m^2)\, K_{\Delta,l}(X|P) = 0 \\ &K_{\Delta,l}(X|P)\Big|_{X\to\bar{P}} \sim \#\,\delta(P-\bar{P}) \\ &K_{\Delta,0} \sim \frac{C_{\Delta,0}}{(-2X\cdot P)^{\Delta}} \sim \int \frac{dt}{t}\, t^{\Delta}\, e^{-2t\,X\cdot P} \end{aligned} \right.$$

What is the bulk dual (position space version) of a CPW?

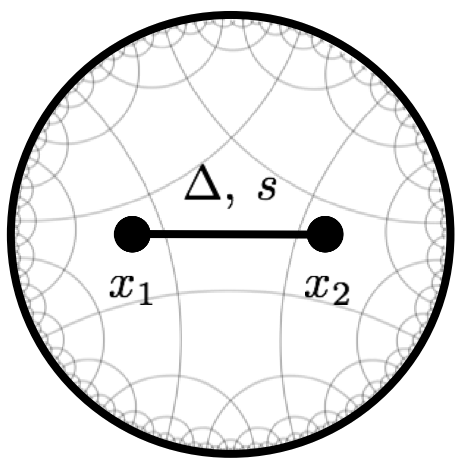
[Ferrara, Grillo, Gatto, Todorov, Fronsdal...]

“Momentum space” for AdS

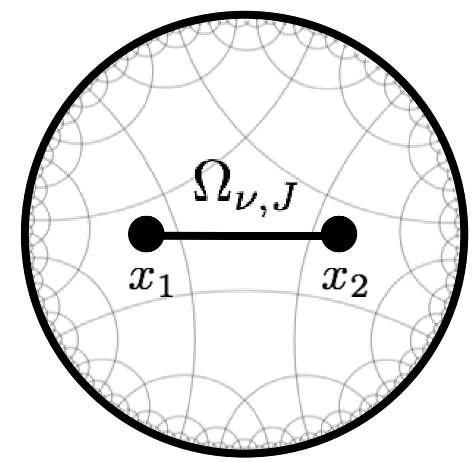
Expand in basis of bi-tensorial harmonic functions (analogue of plane waves):

$$\left[\nabla^2 + \left(\tfrac{d}{2} + i\nu\right)\left(\tfrac{d}{2} - i\nu\right) + J\right]\Omega_{\nu,J} = 0, \qquad \underset{\text{divergence-less}}{\nabla \cdot \Omega_{\nu,J} = 0,} \qquad \underset{\text{trace-less}}{(g \cdot \Omega_{\nu,J}) = 0}$$

Bulk-to-bulk propagators:



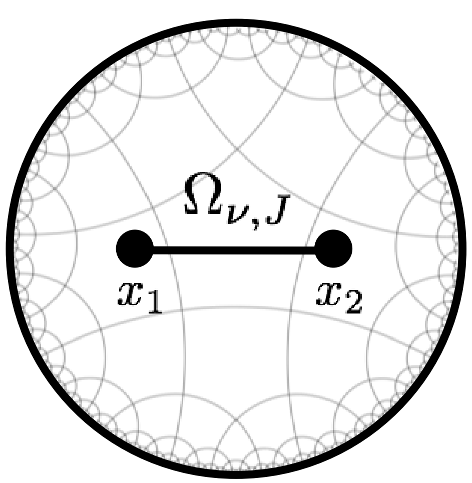
$$= \sum_{J=0}^{s} \int_{-\infty}^{\infty} d\nu \, g_J(\nu)$$



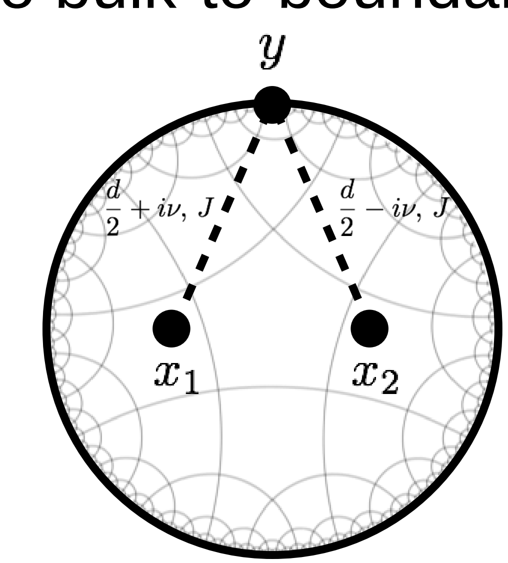
$m^2 R^2 = \Delta(\Delta - d) - s$

[Massive fields: Costa et al.`14, Massless: Bekaert et al. `14; Sleight, M.T. `17]

Harmonic functions factorise into bulk-to-boundary propagators:



$$= \int_{\partial \mathrm{AdS}} d^d y$$



$$= \frac{\nu^2}{\pi} \int dP \, \underbrace{K_{\frac{d}{2}+i\nu}(X_1;P) K_{\frac{d}{2}-i\nu}(X_2;P)}_{e^{ip\cdot(x_1 - x_2)} = e^{ip\cdot x_1} e^{-ip\cdot x_2}}$$

[Leonhardt, Manvelyan, Rühl `03; Costa et al.`14]

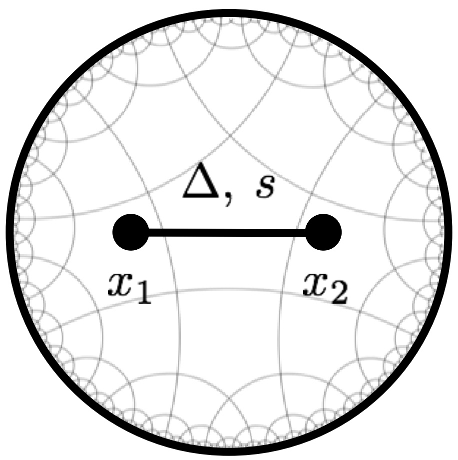
“Momentum space” for AdS

Expand in basis of bi-tensorial harmonic functions (analogue of plane waves):

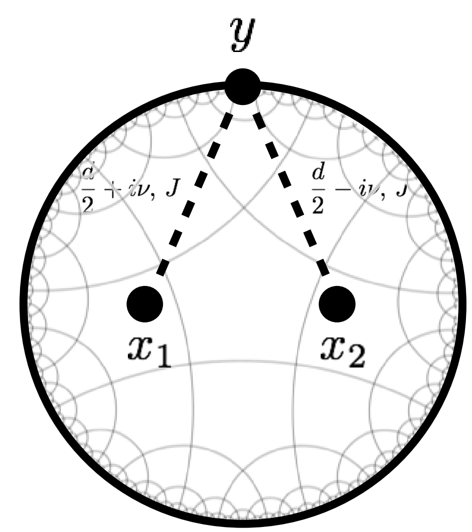
$$\left[\nabla^2 + \left(\tfrac{d}{2} + i\nu\right)\left(\tfrac{d}{2} - i\nu\right) + J\right]\Omega_{\nu,J} = 0, \qquad \nabla \cdot \Omega_{\nu,J} = 0, \qquad (g \cdot \Omega_{\nu,J}) = 0$$

divergence-less   trace-less

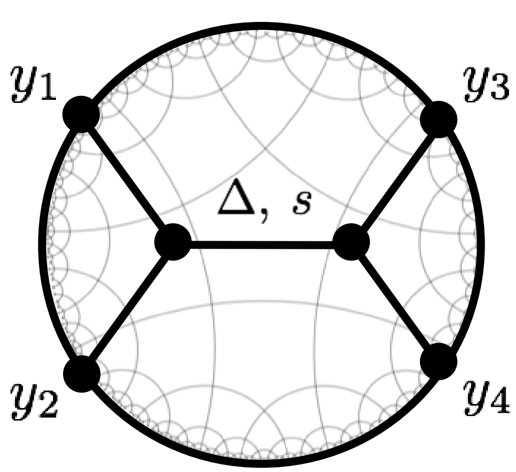
Bulk-to-bulk propagators:



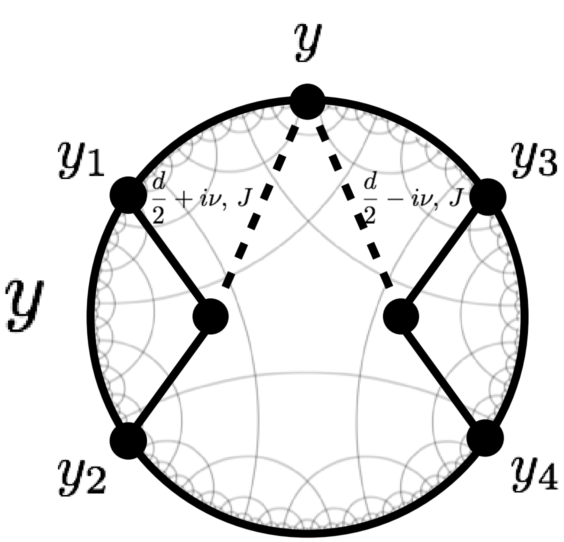
$$= \sum_{J=0}^{s} \int_{-\infty}^{\infty} d\nu\, g_J(\nu) \int_{\partial \mathrm{AdS}} d^d y$$



At **tree-level,** diagrams factorise into **lower-point trees**, which are connected via conformal integration over the boundary:



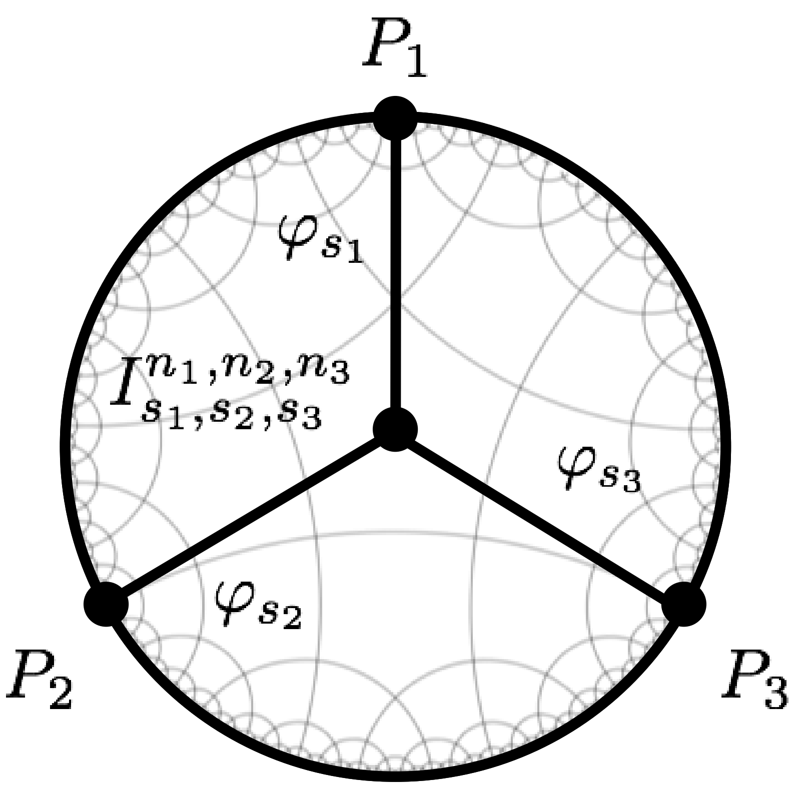
$$= \sum_{J=0}^{s} \int_{-\infty}^{\infty} d\nu\, g_J(\nu) \int_{\partial \mathrm{AdS}} d^d y$$



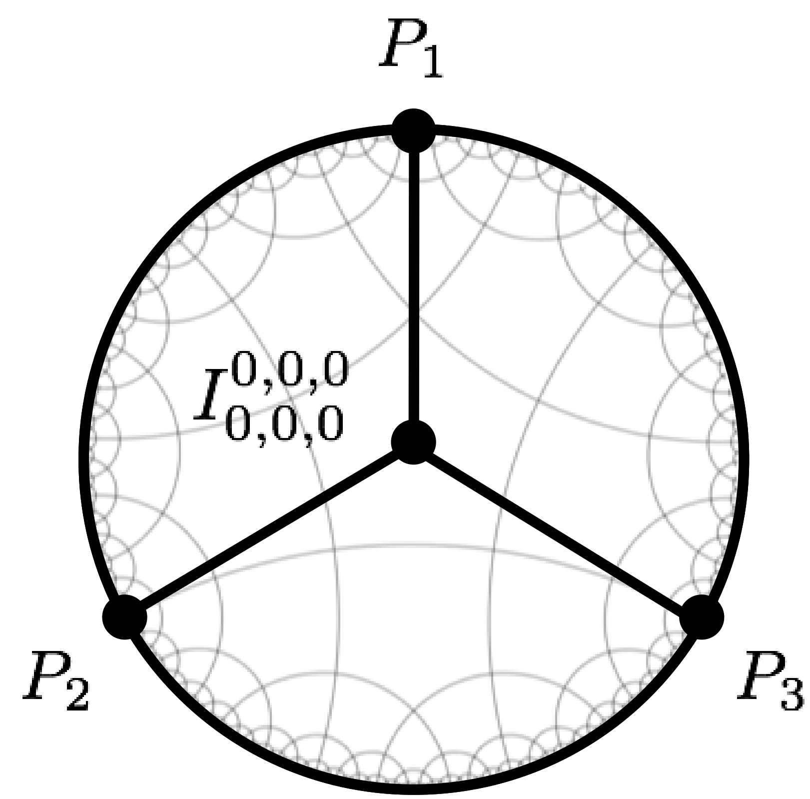
No AdS/CFT assumption but only kinematical rewriting!

“Fourier transforming” 3pt vertices

Standard Trick: Reduce integral over AdS to its scalar seed



$$= \mathcal{F}^{n_1,n_2,n_3}_{s_1,s_2,s_3}\left(P_i, \partial_{P_i}\right)\Big[$$

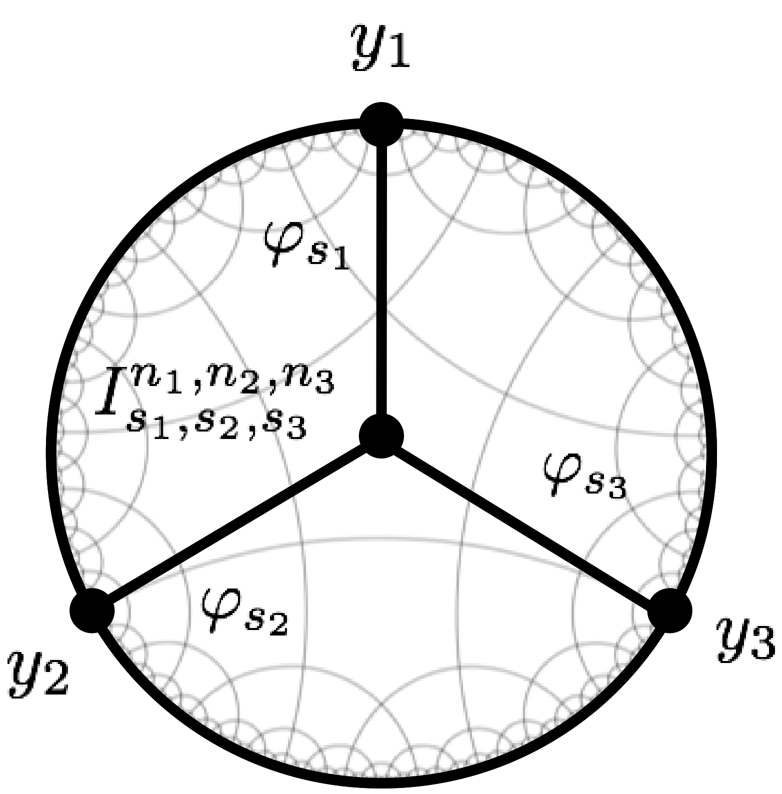


$$\Big]$$

[Mück et al.; Freedman et al. `98]

Spinning tree level 3pt diagrams

Result takes the form:



$$= \sum_{m_i} C^{s_1,s_2,s_3}_{m_1,m_2,m_3} \frac{\mathsf{Y}_1^{s_1-m_2-m_3}\mathsf{Y}_2^{s_2-m_3-m_1}\mathsf{Y}_3^{s_3-m_1-m_2}\mathsf{H}_1^{m_1}\mathsf{H}_2^{m_2}\mathsf{H}_3^{m_3}}{(y_{12})^{\frac{\tau_1+\tau_2-\tau_3}{2}}(y_{23})^{\frac{\tau_2+\tau_3-\tau_1}{2}}(y_{31})^{\frac{\tau_3+\tau_1-\tau_2}{2}}}$$

$$\tau_i = \Delta_i - s_i$$

The above problem suggest a **new basis** for 3pt CFT structures:

$$\delta_{12} = \tfrac{1}{2}(\tau_1+\tau_2-\tau_3)$$

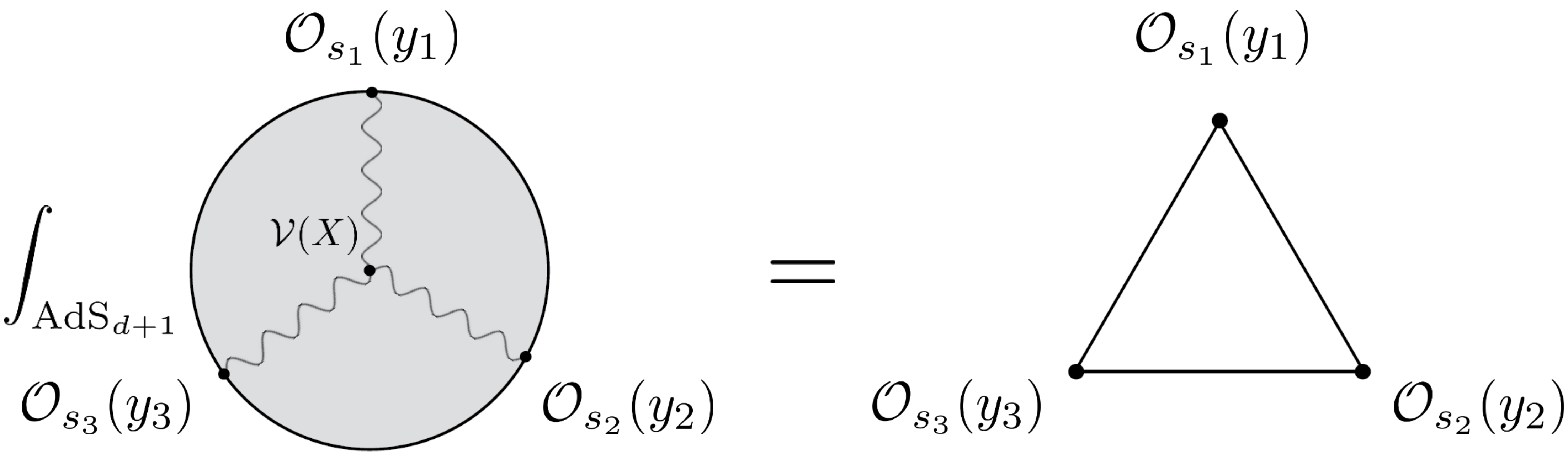
$$\tau = \Delta - s$$

$$[[\mathcal{O}_{\Delta_1,s_1}(y_1)\mathcal{O}_{\Delta_2,s_2}(y_2)\mathcal{O}_{\Delta_3,s_3}(y_3)]]^{(\mathbf{n})} \sim \frac{\mathsf{H}_1^{n_1}\mathsf{H}_2^{n_2}\mathsf{H}_3^{n_3}}{(y_{12})^{\delta_{12}}(y_{23})^{\delta_{23}}(y_{31})^{\delta_{31}}}$$

$$\times \left[\prod_{i=1}^{3} \# J_{\ldots}\left(\sqrt{q_i}\right)\right] \mathsf{Y}_1^{s_1-n_2-n_3}\mathsf{Y}_2^{s_2-n_3-n_1}\mathsf{Y}_3^{s_3-n_1-n_2}$$

$$q_i = H_i \partial_{Y_{i-1}} \partial_{Y_{i+1}}$$

Linear Map for any cubic coupling



We can holographically reconstruct each basis element $[[\mathcal{O}_{\Delta_1,s_1}(x_1)\mathcal{O}_{\Delta_2,s_2}(x_2)\mathcal{O}_{\Delta_3,s_3}(x_3)]]^{(\mathbf{n})}$

$$\mathcal{I}^{n_1,n_2,n_3}_{s_1,s_2,s_3} = \sum_{m_i=0}^{n_i} C^{n_1,n_2,n_3}_{s_1,s_2,s_3;m_1,m_2,m_3}\; I^{m_1,m_2,m_3}_{s_1,s_2,s_3}$$

$$\begin{cases} \delta_{12} = \frac{1}{2}(\tau_1+\tau_2-\tau_3) \\ \tau = \Delta - s \end{cases}$$

$$C^{n_1,n_2,n_3}_{s_1,s_2,s_3;m_1,m_2,m_3} = \left(\frac{d-2(s_1+s_2+s_3-1)-(\tau_1+\tau_2+\tau_3)}{2}\right)_{m_1+m_2+m_3} \prod_{i=1}^{3}\left[2^{m_i}\binom{n_i}{m_i}(n_i+\delta_{(i+1)(i-1)}-1)_{m_i}\right]$$

$$I^{n_1,n_2,n_3}_{s_1,s_2,s_3}(\Phi_i) = \eta^{M_1(n_3)M_2(n_3)}\eta^{M_2(n_1)M_3(n_1)}\eta^{M_3(n_2)M_1(n_2)}(\partial^{N_3(k_3)}\Phi_{M_1(n_2+n_3)N_1(k_1)})$$
$$\times\ (\partial^{N_1(k_1)}\Phi_{M_2(n_3+n_1)N_2(k_2)})(\partial^{N_2(k_2)}\Phi_{M_3(n_1+n_2)N_3(k_3)})$$

Weight Shifting Operators

Cubic couplings induce deformations of gauge transformations and gauge symmetries

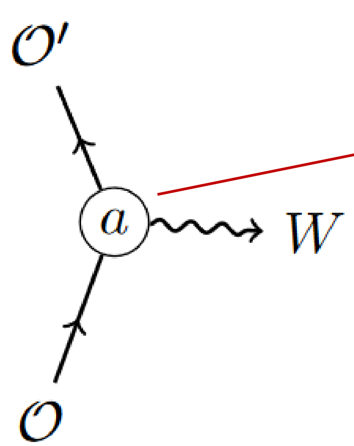
$$\int \left[(\delta^{(1)}\Phi)\Box\Phi + \delta^{(0)}\mathcal{V} \right] = 0$$

The commutator of two gauge transformations closes to the lowest order automatically: extract gauge bracket (field independent)

$$\delta^{(0)}_{[\epsilon_1}\delta^{(1)}_{\epsilon_2]} \approx \delta^{(0)}_{[\![\epsilon_1,\epsilon_2]\!]^{(0)}}$$

The deformation of gauge transformations are the most general conformal differential operators that can be written down!

$$\delta^{(1)}_{\epsilon} \sim \mathcal{D}^{(a)A}$$



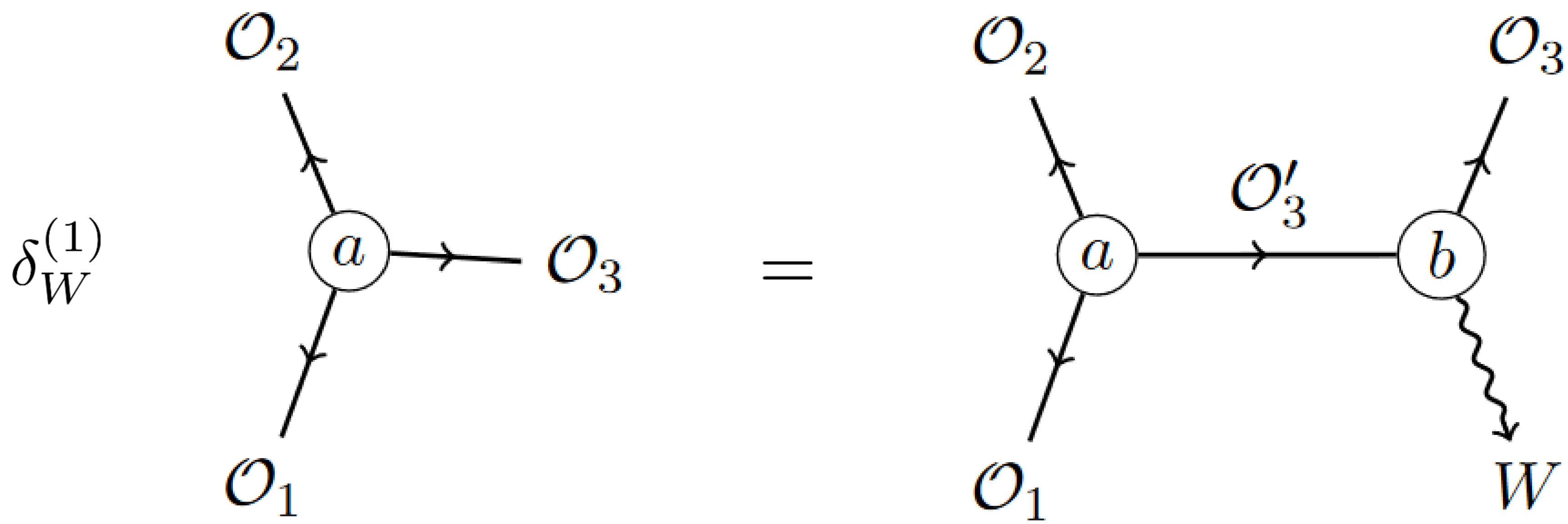
(a) Labels the possible differential operators

Explicit classification known in AdS/CFT

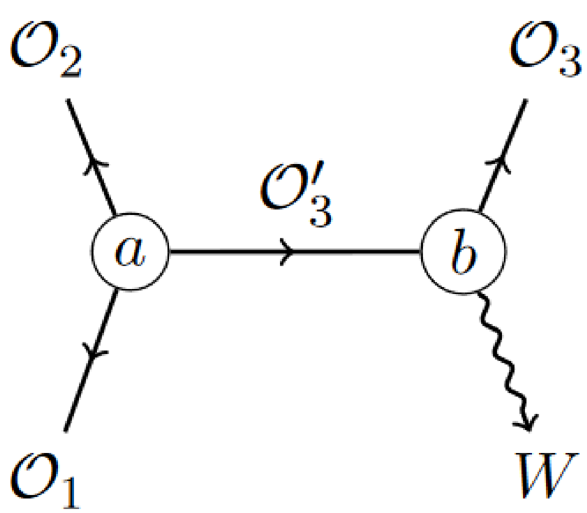
[Joung & MT '13]

Weight Shifting Operators

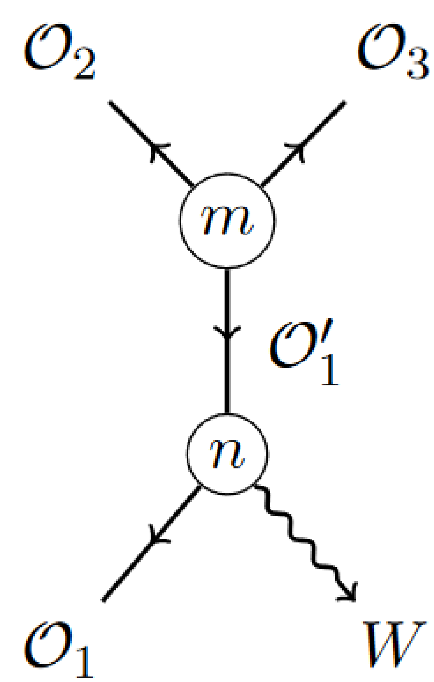
Closure, Jacobi, covariance of cubic couplings can be explicitly written down in terms of 6j symbols of the conformal group:



One obtains crossing relations for HS transformations



$$= \sum_{\mathcal{O}_1', m, n} \begin{Bmatrix} \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_1' \\ \mathcal{O}_3 & W & \mathcal{O}_3' \end{Bmatrix}^{ab}_{mn}$$

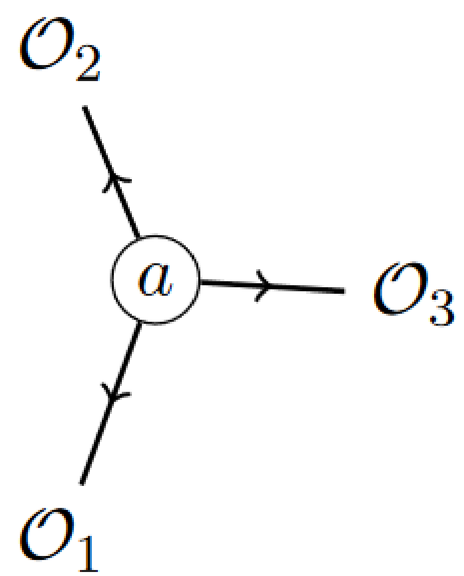


6j symbol

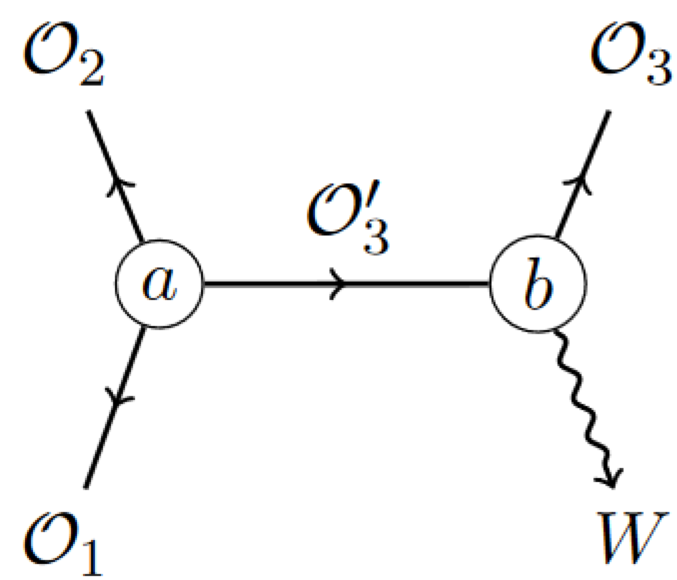
Weight Shifting Operators

Closure, Jacobi, covariance of cubic couplings can be explicitly written down in terms of 6j symbols of the conformal group:

δ_W



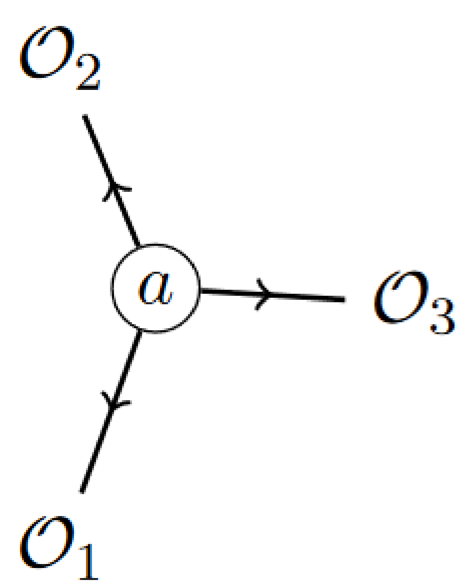
$=$



$+\text{t-ch} + \text{u-ch}$

Noether procedure for cubic vertices at quartic order:

δ_W



$$= \sum g_{\mathbf{n}} g_{\bar{\mathbf{n}}} \begin{Bmatrix} \mathcal{O}' & \mathcal{O}_2 & \mathcal{O}_1 \\ \mathcal{O} & W & \mathcal{O}_3 \end{Bmatrix}^{\mathbf{n},\bar{\mathbf{n}}}_{\mathbf{m},\bar{\mathbf{m}}} = 0$$

Many solutions are known: type A_n, B_n,…

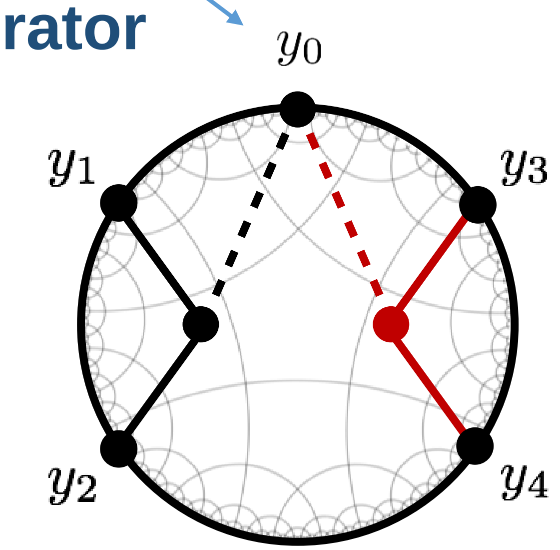
Infinite number of equations but finitely many terms for each equation fix local vertices uniquely

Going to Mellin Space

$$F^{\mathbf{n},\bar{\mathbf{n}}}_{\Delta,l}(y_i) \sim \int d^d y_0 \; [[\mathcal{O}_{\Delta_1,J_1}(y_1)\mathcal{O}_{\Delta_2,J_2}(y_2)\mathcal{O}_{\Delta,l}(y_0)]]^{(\mathbf{n})} \, [[\tilde{\mathcal{O}}_{\Delta,l}(y_0)\mathcal{O}_{\Delta_3,J_3}(y_3)\mathcal{O}_{\Delta_4,J_4}(y_4)]]^{(\bar{\mathbf{n}})}$$

$$\longrightarrow \quad \underbrace{\sum_{r_i}(z_1\cdot\partial_{y_1})^{r_1}(z_2\cdot\partial_{y_2})^{r_2}(z_3\cdot\partial_{y_3})^{r_3}(z_4\cdot\partial_{y_4})^{r_4}}\int d^d y_0 \frac{1}{(y_{01}^2)^{\alpha_1}(y_{02}^2)^{\alpha_2}(y_{03}^2)^{\alpha_3}(y_{04}^2)^{\alpha_4}}$$

The coupling itself knows everything of the differential operator



Everything is reduced to a single scalar integral!

$$\sim \sum_m \frac{\mathcal{Q}^{\mathbf{n},\bar{\mathbf{n}}}_{l,m}(s|W_{ij})}{t-\tau-2m} + \text{shadow}$$

Orthogonality of conformal blocks can be read off from the leading pole, e.g.:

$$\boxed{\mathcal{Q}^{\mathbf{n},\mathbf{0}}_{l,0}(s|W_{ij}) \sim \Upsilon^{(\mathbf{n_1},\mathbf{n_2},\mathbf{n_3};\mathbf{0})}_{\mathbf{J}}(s|W_{ij})\, Q^{(\tau+2n_1,\tau+2n_2,2n_1,2n_2)}_{l-n_1-n_2}(s)}$$

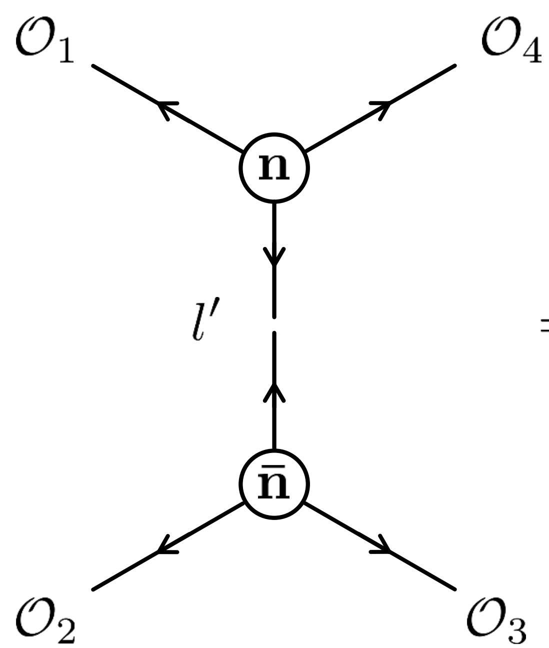
Remarkable Fact: factorization of l dependence from external spin dependence!!

Inversion formulas manifest in terms of the Continuous Hahn polynomial

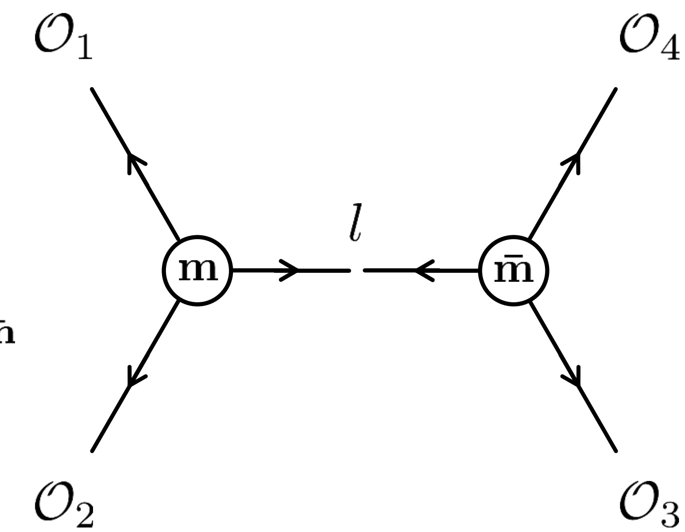
Applications

- Crossing Kernels
- Large N fixed points
- Wilson-Fisher

Crossing Kernels



$$= \sum_{l,\Delta;\mathbf{m},\bar{\mathbf{m}}} \underbrace{\begin{Bmatrix} \mathcal{O}' & \mathcal{O}_2 & \mathcal{O}_1 \\ \mathcal{O} & \mathcal{O}_4 & \mathcal{O}_3 \end{Bmatrix}^{\mathbf{n},\bar{\mathbf{n}}}_{\mathbf{m},\bar{\mathbf{m}}}}$$



Crossing Kernel/Racah Coefficient/6j symbol

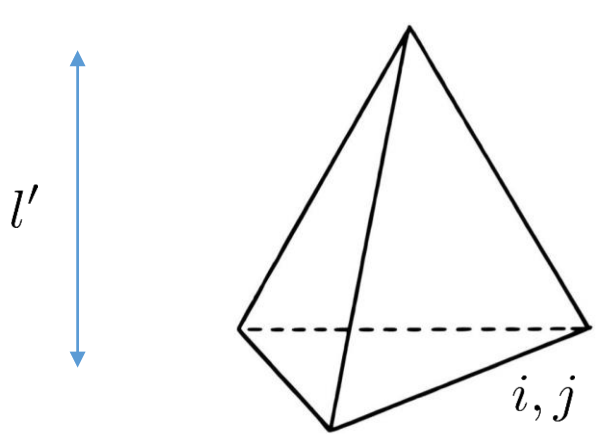
Arbitrary exchanged spin (single structure): $W_{12}^{J_1} W_{21}^{J_2} W_{34}^{J_3} W_{43}^{J_4}$

$$\begin{Bmatrix} \mathcal{O}' & \mathcal{O}_2 & \mathcal{O}_1 \\ \mathcal{O} & \mathcal{O}_4 & \mathcal{O}_3 \end{Bmatrix}^{\mathbf{n},\bar{\mathbf{n}}}_{\mathbf{m},\bar{\mathbf{m}}} \sim \sum_{p=0}^{l'} \sum_{i+j=0}^{l'-p}$$

Tetrahedral sum

$$\times\ {}_4F_3 \left(\begin{matrix} -j, 1-i-j-p-\frac{\tau}{2}, 1+i+p-\frac{d-\tau}{2}, \Delta-\frac{t}{2} \\ \frac{t-2\Delta+2}{2}-j, i+p-l'+\frac{d-\tau}{2}, l'-i-j-p+\frac{\tau}{2} \end{matrix} ; 1 \right)$$

$$\times\ {}_4F_3 \left(\begin{matrix} p-l, l+p+t-1, \frac{d-\tau}{2}-i+\frac{t-2\Delta}{2}, l'-i+\frac{t-2\Delta}{2}+\frac{\tau}{2} \\ p+\frac{t}{2}, p+\frac{t}{2}, \frac{d}{2}-2\Delta+t \end{matrix} ; 1 \right)$$



[C. Sleight & M.T.]

Mean Field Theory

The first step is to extract the leading order OPE

$$\mathcal{A}^{(0)}_{0000} = \left[1 + u^{\Delta} + \left(\frac{u}{v}\right)^{\Delta}\right] = 1 + \sum_{l,q=0}^{\infty} {}^{(0)}a^{[\Phi\Phi]}_{q,l} u^{\Delta+q} g_{2\Delta+2q,l}(u,v)$$

A simple test for inversion formula but we need to go to Mellin space…

$$\int_0^{\infty} dx\, x^{s-1}\, {\color{red}x^{\Delta}} \sim ?$$

This integral is divergent…

Mean Field Theory

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$$\mathcal{A}^{(0)}_{0000} = \left[1 + u^{\Delta} + \left(\frac{u}{v}\right)^{\Delta}\right] = 1 + \sum_{l,q=0}^{\infty} {}^{(0)}a^{[\Phi\Phi]}_{q,l}\, u^{\Delta+q} g_{2\Delta+2q,l}(u,v)$$

A simple test for inversion formula but we need to go to Mellin space…

[(a lot of papers in feynman diagram community), M.T. 2016; Bekaert et al. 2016]

$$\left(\int e^{ipx_1} e^{ipx_2} \sim \delta(x_1 + x_2)\right) \qquad \int_0^{\infty} dx\, x^{s-1}\, {\color{red} x^{\Delta}} \sim \langle s + \Delta \rangle$$

This integral is divergent… (as for HS theory in flat space)

Can still define it as a distribution upon considering it as a functional on an appropriate space of functions

$$\int_{-i\infty}^{+i\infty} \frac{ds}{4\pi i} x^{-s} f(s) \langle s + \Delta \rangle = x^{\Delta} f(-\Delta)$$

The Mellin transform of Wick-contractions is a delta-function distribution

O(N) model

The O(N) model is not much different than MFT

$$u^{\Delta/2} + \left(\frac{u}{v}\right)^{\Delta/2} + u^{\Delta/2}\left(\frac{u}{v}\right)^{\Delta/2}$$

$$\sum_{l}^{\infty} {}^{(0)}a_l^{[\mathcal{J}]} u^{(d-2)/2} g_{d-2/2,l}\left(u,v\right) \qquad \sum_{q}\sum_{l}^{\infty} {}^{(0)}a_{l,q}^{[OO]} u^{(d-2+2q)/2} g_{(d-2+2q)/2,l}\left(u,v\right)$$

The above conformal block expansion can be arranged in twist block expansions

N.B. The above sum are not uniformly convergent:

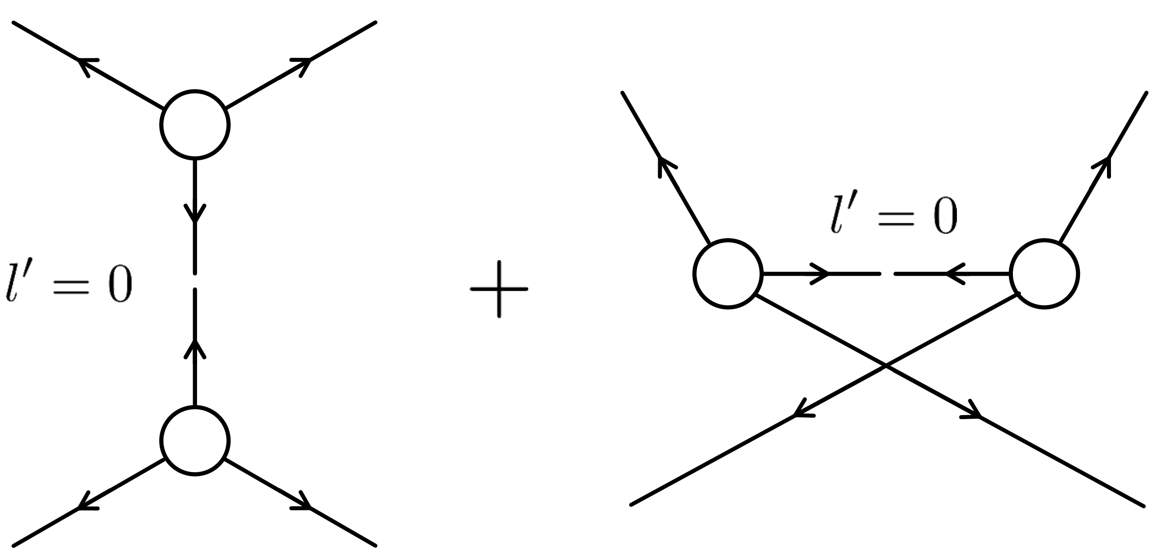
$$\sum_{l}^{\infty} {}^{(0)}a_l^{[\mathcal{J}]} u^{(d-2)/2} g_{d-2/2,l}\left(u,v\right) = u^{(d-2)/2}\left(1 + v^{-(d-2)/2}\right) + \underbrace{\left(\sum_l g_l\, a_l^{[\mathcal{J}]}\right)}_{=0} u^{(d-2)/2+1} + \dots$$

Sum over spin must reproduce singularities in the crossed channels…

Anomalous Dimensions

The simplest external scalar case:

$$[\mathcal{O}\mathcal{O}]_{l,n} \begin{cases} n = 0 & \text{Leading twist operators} \\ n > 0 & \text{Subleading twist operators} \end{cases}$$



We can easily obtain the result for scalar double-trace deformations

$$\delta\gamma_{0,0}^{[\Phi\Phi]} = \frac{2\,\Gamma(\tau)\Gamma\left(\frac{d-\tau}{2}\right)^2}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{\tau}{2}\right)^2\Gamma\left(\frac{d}{2}-\tau\right)}\, c_{\Phi\Phi\mathcal{O}}\,,$$

$$\delta\gamma_{0,l}^{[\Phi\Phi]} = \left(\frac{1+(-1)^l}{2}\right)\,\delta\gamma_{0,0}\;{}_4F_3\left(\begin{matrix}-l,\,2\Delta+l-1,\,\frac{d-\tau}{2},\,\frac{\tau}{2}\\ \frac{d}{2},\,\Delta,\,\Delta\end{matrix};1\right)$$

Anomalous Dimensions

The simplest external scalar case:

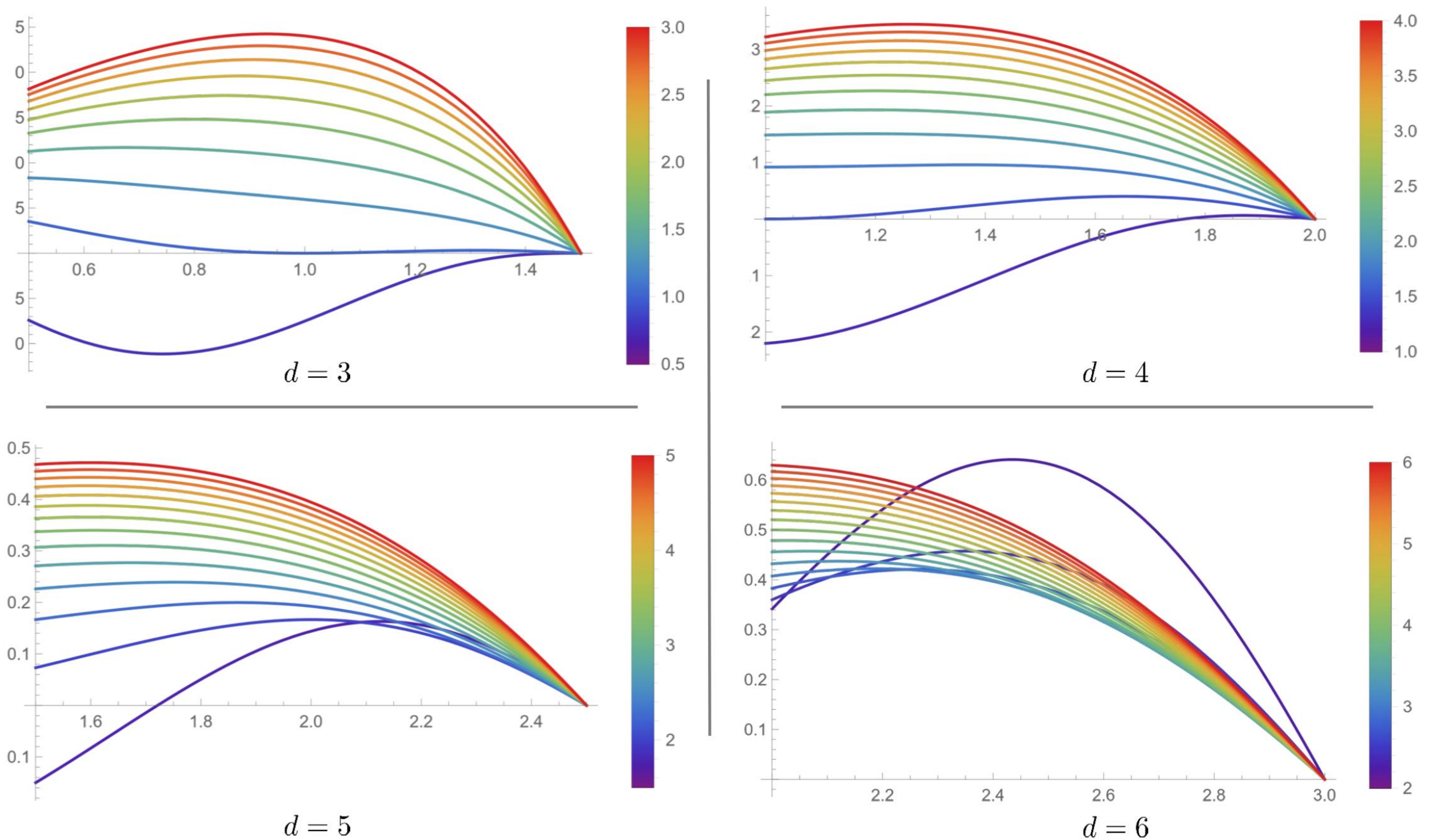
$$[\mathcal{O}\mathcal{O}]_{l,n} \quad \begin{cases} n = 0 & \text{Leading twist operators} \\ n > 0 & \text{Subleading twist operators} \end{cases}$$

We obtain explicit expressions for all subleading twist double trace operators

$$\boxed{\gamma_{n,l} \sim \sum_{j=0}^{n} D_j T^n_{n-j,j}}$$

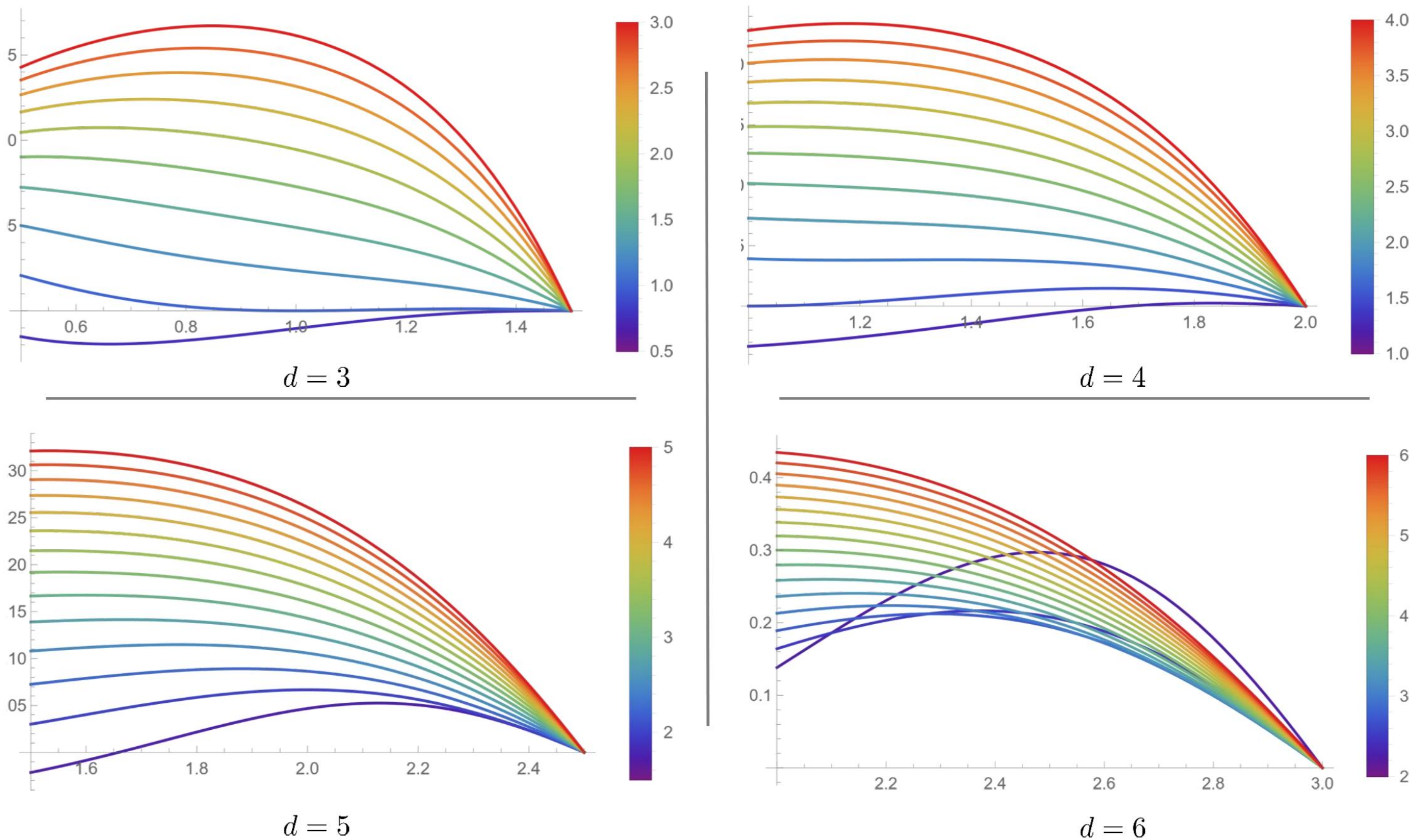
$$\begin{aligned} T^n_{ij} &= \int_{-i\infty}^{i\infty} \frac{ds}{4\pi i}\,\Gamma(-\tfrac{s}{2})^2\Gamma(\tfrac{d+s-\tau}{2}+i)\Gamma(\tfrac{s+\tau}{2}+j)\,Q_l^{2\Delta+2n,2\Delta+2n,0,0}(s) \\ &= \frac{2^l\Gamma\left(\frac{2j+\tau}{2}\right)^2\Gamma\left(\frac{d+2i-\tau}{2}\right)^2\left(\frac{2\Delta+2n}{2}\right)_l^2}{(l+2\Delta+2n-1)_l\Gamma\left(\frac{d+2i+2j}{2}\right)}\,{}_4F_3\left(\begin{matrix}-l, 2\Delta+2n+l+1, \frac{d}{2}+i-\frac{\tau}{2}, j+\frac{\tau}{2} \\ \frac{d}{2}+i+j, \Delta+n, \Delta+n\end{matrix};1\right) \end{aligned}$$

$[OO]_{1,0}$



On the real axis the dimension of the CPW in t, u channel. The bar is the dimension of the external legs

$[OO]_{2,0}$



On the real axis the dimension of the CPW in t, u channel. The bar is the dimension of the external legs

Wilson-Fisher

The simplest external scalar case:

$$[\mathcal{O}\mathcal{O}]_{l,n} \quad \begin{cases} n = 0 & \text{Leading twist operators} \\ n > 0 & \text{Subleading twist operators} \end{cases}$$

We obtain a closed formula for arbitrary l and n: $d = 4 - \epsilon$ $\quad \tau = 2 - \epsilon$

$$\boxed{\delta\gamma_{n,l} = \epsilon\, c_{\Phi\Phi\mathcal{O}}\, (-1)^l \frac{(\Delta-1)^2}{(\Delta+n-1)^2}\, {}_4F_3\left(\begin{matrix} 1, 1, -l, l+2\Delta+2n-1 \\ 2, \Delta+n, \Delta+n \end{matrix}; 1\right)}$$

The above result applies to the WF-fixed point with: $\lambda \int \mathcal{O}^2$

$$\langle \Phi\bar{\Phi}\Phi\bar{\Phi} \rangle$$

$$\Phi = \phi\phi \qquad \bar{\Phi} = \bar{\phi}\bar{\phi} \qquad \mathcal{O} = \phi\bar{\phi} \qquad c_{\Phi\bar{\Phi}\mathcal{O}} = \frac{4}{N}$$

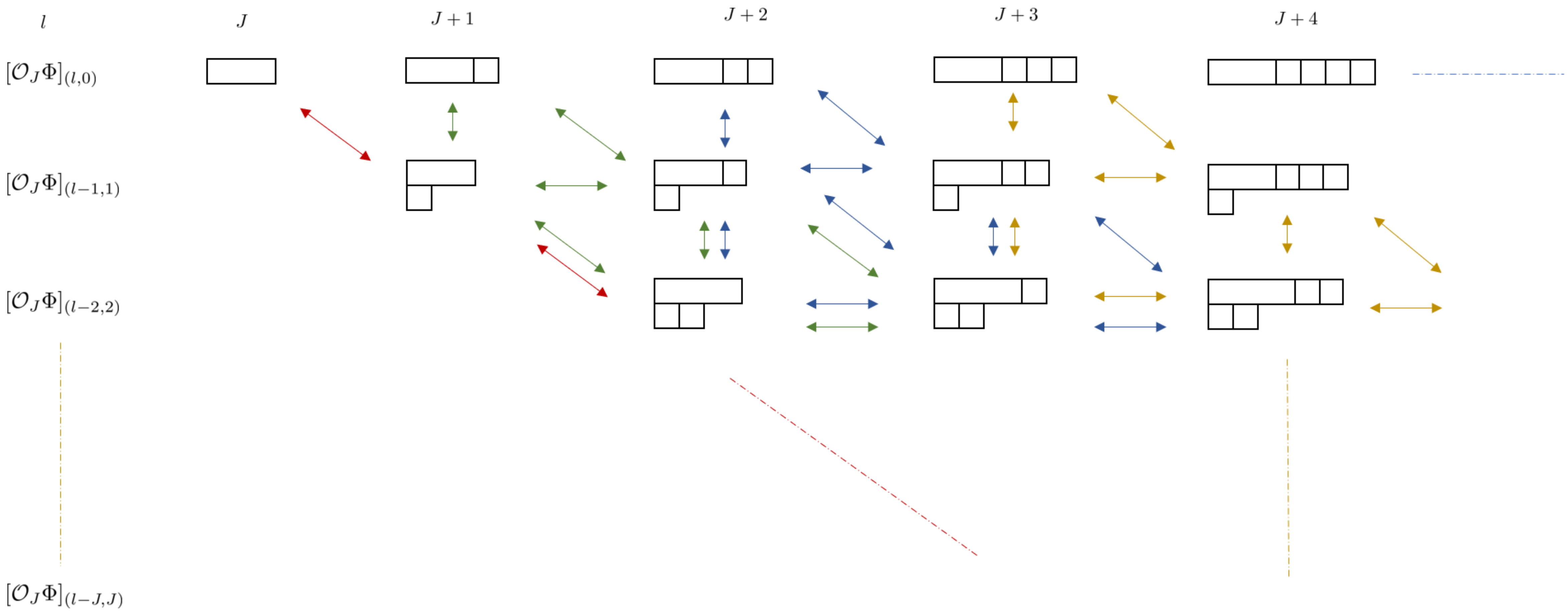
Large spin behavior same independently on n:

$$\gamma^{[\Phi\Phi]}_{n,l\to\infty} \sim \frac{8}{N}\,\frac{\log l}{l^2}\,\epsilon$$

JOJO

$$\langle \mathcal{O}_J(y_1)\Phi(y_2)\mathcal{O}_J(y_3)\Phi(y_4)\rangle$$

MFT: $(2\,W_{13}W_{31})^J\,u^{\tau_1+\tau_2}$

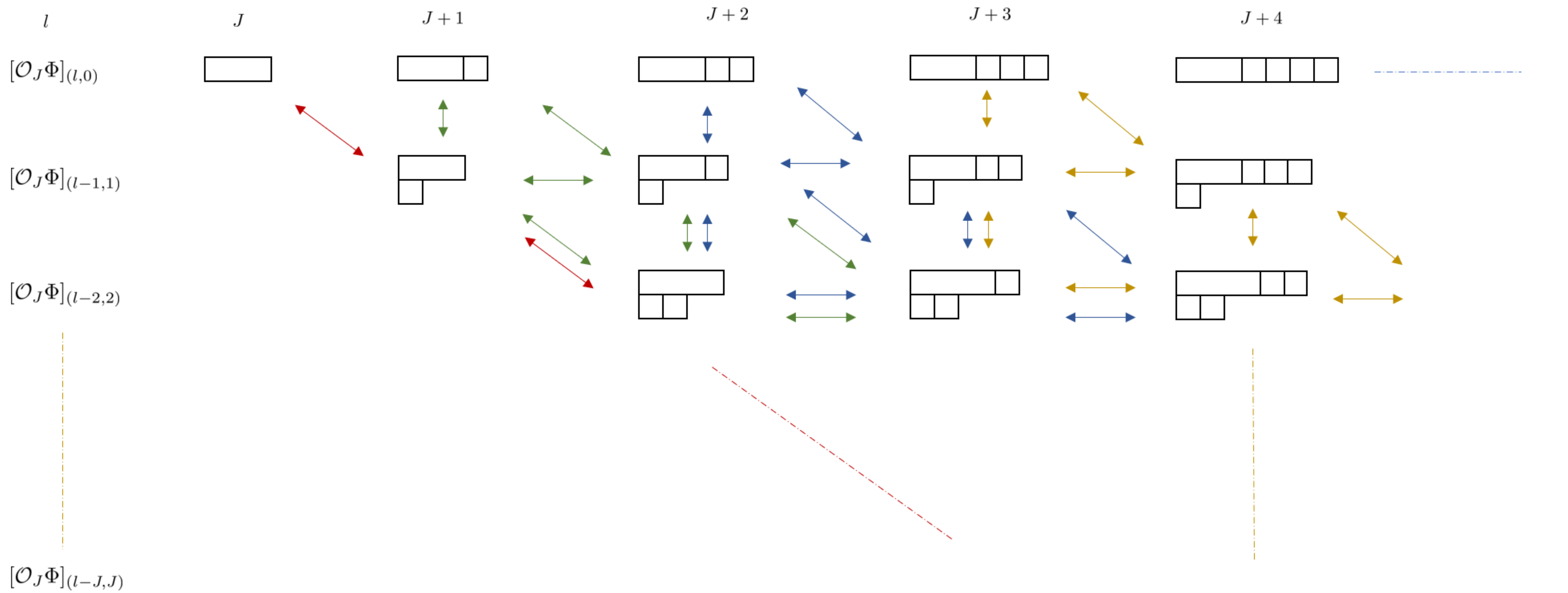


First the OPE:

$$a_{(l,0)}^{[\mathcal{O}_J\Phi]} = \frac{2^{l-J}(2J+\tau_1)_{l-J}(\tau_2)_{l-J}}{(l-J)!(l+J+\tau_1+\tau_2-1)_{l-J}}$$

JOJO

$$\langle \mathcal{O}_J(y_1)\Phi(y_2)\mathcal{O}_J(y_3)\Phi(y_4)\rangle \quad \Rightarrow \quad \textbf{MFT:} \quad (2\,W_{13}W_{31})^J\, u^{\tau_1+\tau_2}$$



And then anomalous dimensions:

$$\delta\gamma^{[\mathcal{O}_J\Phi]}_{(J,0)} = \frac{J!}{(-2)^J}\frac{\left(\frac{d-\tau+\tau_1+\tau_2}{2}+J-1\right)_J}{\left(\frac{\tau_1+\tau_2-\tau}{2}\right)_J} \times \frac{2\Gamma(\tau)\Gamma\left(\frac{d-\tau+\tau_1-\tau_2}{2}\right)\Gamma\left(\frac{d-\tau-\tau_1+\tau_2}{2}\right)}{\Gamma\left(\frac{d}{2}+1\right)\Gamma\left(\frac{d}{2}-\tau\right)\Gamma\left(\frac{\tau+\tau_1-\tau_2}{2}\right)\Gamma\left(\frac{\tau-\tau_1+\tau_2}{2}\right)}\, c_{\mathcal{O}_J\mathcal{O}_J\mathcal{O}}c_{\mathcal{O}\Phi\Phi}$$

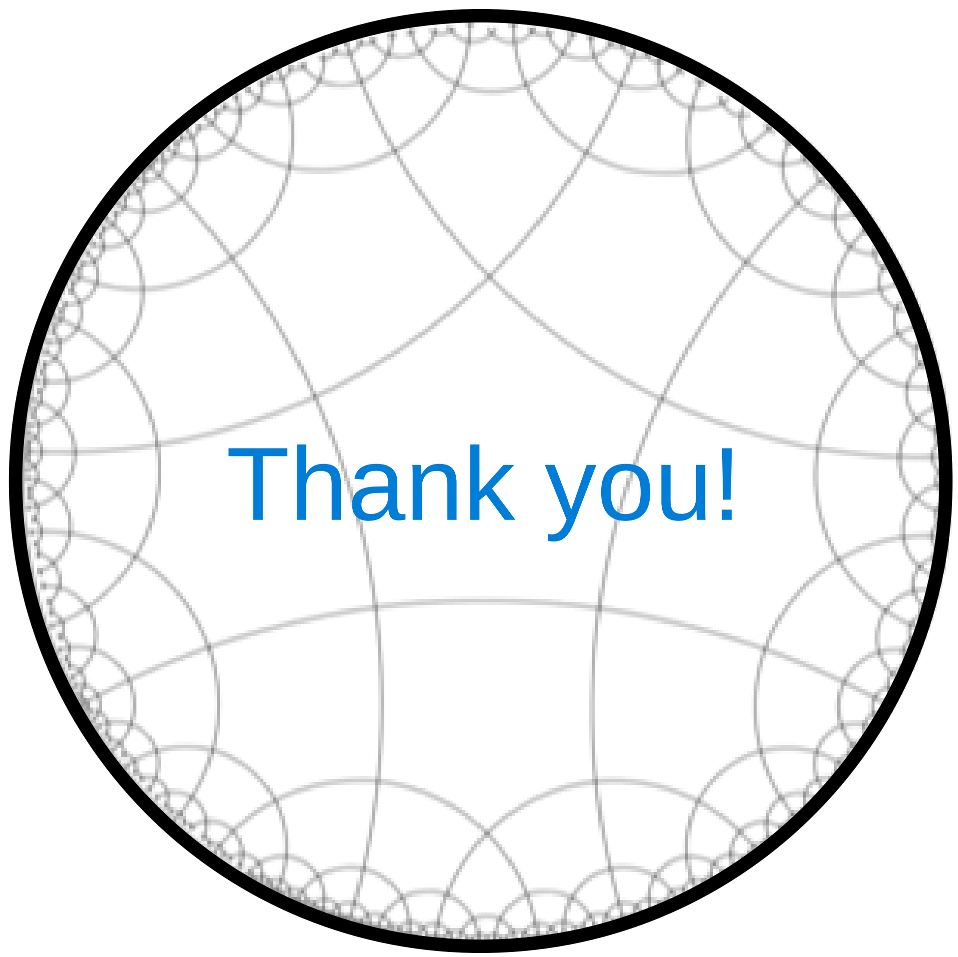
Outlook

- We barely scratched the surface of a remarkable hidden structure behind CFT conformal blocks with spinning external and internal legs!

- Mellin space makes manifest inversion formulas and reduces them to finite dimensional linear algebra

- Lesson: The bulk to boundary explicit map of arXiv:1702.08619 can teach us a lot about the spinning bootstrap and it is the analogues of momentum space for flat space HS correlators

- Analyticity in spin sets the convergence rate of quartic interactions (EFT & 1/Box)


Thank you!